I.6.2. Curl and divergence

Show that the curl and the divergence of a vector field transform as a pseudovector field and a scalar field, respectively.

(3 points)

Solution

Consider the curl of a vector field v, as defined in § 6.2:

$$c^i(x) = \epsilon^{ijk}\partial_j v_k(x)$$

The transformed curl is

$$\begin{aligned}
\tilde{c}^i(\tilde{x}) &= \tilde{\epsilon}^{ijk}\tilde{\partial}_j v_k(x) \\
&= \tilde{\epsilon}^{ijk}\left(D^{-1}\right)^{\ell}{}_{j}\,\partial_\ell\left(D^{-1}\right)^{m}{}_{k}\,v_m(x) && \text{by Problem I.6.1} \\
&= \delta^i{}_n\,\tilde{\epsilon}^{njk}\left(D^{-1}\right)^{\ell}{}_{j}\left(D^{-1}\right)^{m}{}_{k}\,\partial_\ell\,v_m(x) \\
&= D^i{}_p\left(D^{-1}\right)^{p}{}_{n}\,\tilde{\epsilon}^{njk}\left(D^{-1}\right)^{\ell}{}_{j}\left(D^{-1}\right)^{m}{}_{k}\,\partial_\ell\,v_m(x) \\
&= D^i{}_p\left(D^{-1}\right)^{p}{}_{n}\left(D^{-1}\right)^{\ell}{}_{j}\left(D^{-1}\right)^{m}{}_{k}\,\tilde{\epsilon}^{njk}\partial_\ell\,v_m(x) \\
&= D^i{}_p\,(\det D)\,\epsilon^{p\ell m}\partial_\ell\,v_m(x) && \text{by § 6.1 Remark 9} \\
&= (\det D)D^i{}_p\,c^p(x)
\end{aligned}$$

$\Rightarrow c^i(x)$ transforms as a pseudovector field. 2pt

Now consider the divergence

$$d(x) = \partial_i v^i(x)$$

The transformed divergence is

$$\begin{aligned}
\tilde{d}^i(\tilde{x}) &= \tilde{\partial}_i\tilde{v}^i(\tilde{x}) = \left((D^{-1})^T\right)_i{}^j\,\partial_j D^i{}_k v^k(x) \\
&= \left(D^T\right)_k{}^i\left((D^T)^{-1}\right)_i{}^j\,\partial_j v^k(x) \\
&= \delta_k{}^j\partial_j v^k(x) = \partial_k v^k(x)
\end{aligned}$$

$\Rightarrow d(x)$ transforms as a scalar field. 1 pt