I.6.1. Transformations of tensor fields

a) Consider a covariant rank-N tensor field $t_{i_1 \ldots i_N}(x)$ and find its transformation law under normal coordinate transformations that is analogous to Definition I.6.1; i.e., find how $\tilde{t}_{i_1 \ldots i_N}(\tilde{x})$ is related to $t_{i_1 \ldots i_N}(x)$.

b) Convince yourself that your result is consistent with the transformation properties of (i) a covector x_i (the case $N = 1$), and (ii) the covariant components of the metric tensor g_{ij}.

(4 points)

Solution

a) There are various ways to do this. One option is to start with the transformation property of contravariant tensor fields in Def. I.6.1 and use the metric tensor to lower the indices:

$$\begin{aligned}
\tilde{t}_{i_1 \ldots i_N}(\tilde{x}) &= \tilde{g}_{i_1 j_1} \cdots \tilde{g}_{i_N j_N} \tilde{t}^{j_1 \ldots j_N}(\tilde{x}) \\
&= g_{i_1 j_1} \cdots g_{i_N j_N}\, D^{j_1}_{\ k_1} \cdots D^{j_N}_{\ k_N}\, t^{k_1 \ldots k_N}(x) && \text{by } \tilde{g} = g \text{ and Eqn. (I.6.1)} \\
&= g_{i_1 j_1} \cdots g_{i_N j_N}\, D^{j_1}_{\ k_1} \cdots D^{j_N}_{\ k_N}\, g^{k_1 \ell_1} \cdots g^{k_N \ell_N}\, t_{\ell_1 \ldots \ell_N} \\
&= (gD)_{i_1 k_1} \cdots (gD)_{i_N k_N}\, g^{k_1 \ell_1} \cdots g^{k_N \ell_N}\, t_{\ell_1 \ldots \ell_N} \\
&= \left((D^T)^{-1} g\right)_{i_1 k_1} \cdots \left((D^T)^{-1} g\right)_{i_N k_N}\, g^{k_1 \ell_1} \cdots g^{k_N \ell_N}\, t_{\ell_1 \ldots \ell_N} && \text{by Def. I.5.7} \\
&= g_{m_1 k_1} g^{k_1 \ell_1} \cdots \left((D^T)^{-1}\right)_{i_N}^{\ \ m_N} g_{m_1 k_1} g^{k_N \ell_N}\, t_{\ell_1 \ldots \ell_N} \\
&= \left((D^T)^{-1}\right)_{i_1}^{\ \ m_1} \delta_{m_1}^{\ \ \ell_1} \cdots \left((D^T)^{-1}\right)_{i_N}^{\ \ m_N} \delta_{m_N}^{\ \ \ell_N}\, t_{\ell_1 \ldots \ell_N} && \text{by Def. I.5.1} \\
&= \left((D^T)^{-1}\right)_{i_1}^{\ \ j_1} \cdots \left((D^T)^{-1}\right)_{i_N}^{\ \ j_1} t_{\mathrm{j}_1 \ldots \mathrm{j}_N}
\end{aligned}$$

That is, if the contravariant components of t transform via D, then the covariant ones transform via $(D^T)^{-1}$. 2pts

b) Now consider the special case $N = 1$. From part a) we have

$$\tilde{x}_i = \left((D^T)^{-1}\right)_i^{\ j} x_j$$

and therefore

$$\tilde{x}^i = g^{ij} \tilde{x}_j = g^{ij} \left((D^T)^{-1}\right)_j^{\ k} x_k = \left(g (D^T)^{-1}\right)^{ik} x_k = \left(g (D^T)^{-1}\right)^{ik} g_{k\ell} x^\ell = \left(g (D^T)^{-1} g\right)^i_{\ \ell} x^\ell \qquad (*)$$

But from Def. I.5.7 we have

$$\left(g (D^T)^{-1} g\right)^i_{\ j} = (ggD)^i_{\ j} = g^i_{\ k} g^k_{\ \ell} D^\ell_{\ j} = \delta^i_{\ \ell} D^\ell_{\ j} = D^i_{\ j}$$

and hence $(*)$ is consistent with Proposition I.5.4. 1 pt

For the metric tensor we have

$$\begin{aligned}
\tilde{g}_{ij} &= \left((D^T)^{-1}\right)_i^{\ k} \left((D^T)^{-1}\right)_j^{\ \ell} g_{k\ell} = \left((D^T)^{-1} g D^{-1}\right)_{ij} = \left((D^{-1})^T g D^{-1}\right)_{ij} \\
&= g_{ij} && \text{by Def. I.5.7}
\end{aligned}$$

where we have used Lemma I.5.2. So the result of part a) is consistent with the invariance of the metric. 1pt