I.5.3. Special Lorentz transformations in M_4

Consider the Minkowski space M_4.

a) Show that the following transformations are Lorentz transformations:

i) $D^\mu{}_\nu = \begin{pmatrix} 1 & 0 \\ 0 & R^i{}_j \end{pmatrix} \equiv R^\mu{}_\nu$ (rotations)

where $R^i{}_j$ is any Euclidian orthogonal transformation.

ii) $D^\mu{}_\nu = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv B^\mu{}_\nu$ (Lorentz boost along the x-direction)

with $\alpha \in \mathbb{R}$.

iii) $D^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv P^\mu{}_\nu$ (parity)

iv) $D^\mu{}_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv T^\mu{}_\nu$ (time reversal)

b) Let L be the group of all Lorentz transformations. Show that the rotations defined in part a) i) are a subgroup of L, and so are the Lorentz boosts defined in part a) ii).

c) Let $I^\mu{}_\nu = \delta^\mu{}_\nu$ be the identity transformation. Show that the sets $\{I, P\}$, $\{I, T\}$, and $\{I, P, T, PT\}$ are subgroups of L.

(4 points)

Solution

I.5.3 a) i) $\underline{R^\mu{}_\alpha g_{\mu\nu} R^\nu{}_\lambda} = \left(\begin{array}{c|c}1&0\\ \hline 0&R^T\end{array}\right)\left(\begin{array}{c|c}1&0\\ \hline 0&-1\end{array}\right)\left(\begin{array}{c|c}1&0\\ \hline 0&R\end{array}\right) = \left(\begin{array}{c|c}1&0\\ \hline 0&R^T\end{array}\right)\left(\begin{array}{c|c}1&0\\ \hline 0&-R\end{array}\right)$

$= \left(\begin{array}{c|c}1&0\\ \hline 0&-R^TR\end{array}\right) = \left(\begin{array}{c|c}1&0\\ \hline 0&-\mathbb{1}\end{array}\right) = \underline{g_{\alpha\lambda}}$ ✓

ii) $\underline{B^\mu{}_\alpha g_{\mu\nu} B^\nu{}_\lambda} = \begin{pmatrix}\cosh\alpha&0&0&\sinh\alpha\\0&1&0&0\\0&0&1&0\\\sinh\alpha&0&0&\cosh\alpha\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$

$\times\begin{pmatrix}\cosh\alpha&0&0&\sinh\alpha\\0&1&0&0\\0&0&1&0\\\sinh\alpha&0&0&\cosh\alpha\end{pmatrix}$

$= \begin{pmatrix}\cosh^2\alpha-\sinh^2\alpha&0&0&\cosh\alpha\sinh\alpha-\sinh\alpha\cosh\alpha\\0&-1&0&0\\0&0&-1&0\\\sinh\alpha\cosh\alpha-\cosh\alpha\sinh\alpha&0&0&\sinh^2\alpha-\cosh^2\alpha\end{pmatrix}$

1 $= \begin{pmatrix}1&&&0\\&-1&&\\&&-1&\\0&&&-1\end{pmatrix} = \underline{g_{\alpha\lambda}}$ ✓

iii) $\underline{P^\mu{}_\alpha g_{\mu\nu} P^\nu{}_\lambda} = \left(\begin{array}{c|c}1&0\\ \hline 0&-1\end{array}\right)\left(\begin{array}{c|c}1&0\\ \hline 0&-1\end{array}\right)\left(\begin{array}{c|c}1&0\\ \hline 0&-1\end{array}\right) = \left(\begin{array}{c|c}1&0\\ \hline 0&-1\end{array}\right) = \underline{g_{\alpha\lambda}}$ ✓

1 iv) $\underline{T^\mu{}_\alpha g_{\mu\nu} T^\nu{}_\lambda} = (-)^2 P^\mu{}_\alpha g_{\mu\nu} P^\nu{}_\lambda = \underline{g_{\alpha\lambda}}$ ✓

1 b) $R^\mu{}_\nu$ leaves the time coordinates invariant, and we know that the $R^i{}_j$ form a group (see PHYS 611) → $\{R^i{}_j\}$ is a subgroup of L.
The Lorentz boosts $B^\mu{}_\nu$ are a subgroup of L according to Problem 1.8.

1 c) group tables:

	$\mathbb{1}$	P
$\mathbb{1}$	$\mathbb{1}$	P
P	P	$\mathbb{1}$

and

	$\mathbb{1}$	T
$\mathbb{1}$	$\mathbb{1}$	T
T	T	$\mathbb{1}$

→ $\{\mathbb{1}, P\}$, and $\{\mathbb{1}, T\}$ are subgroups of L under matrix multiplication

	$\mathbb{1}$	P	T	PT
$\mathbb{1}$	$\mathbb{1}$	P	T	PT
P	P	$\mathbb{1}$	PT	T
T	T	PT	$\mathbb{1}$	P
PT	PT	T	P	$\mathbb{1}$

→ $\{\mathbb{1}, P, T, PT\}$ is also a subgroup of L