I.5.1. Lorentz transformations in M_2

Consider the 2-dimensional Minkowski space M_2 with metric $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and 2×2 matrix representations of the pseudo-orthogonal group $O(1,1)$ that leaves g invariant.

a) Let $\sigma, \tau = \pm 1$, and $\phi \in \mathbb{R}$. Show that any element of $O(1,1)$ can be written in the form

$$D_{\sigma,\tau}(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$$

To study $O(1,1)$ it thus suffices to study the matrices $D(\phi) := D_{+1,+1} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}$.

b) Show explicitly that the set $\{D(\phi)\}$ forms a group under matrix multiplication (which is a subgroup of $O(1,1)$ that is sometimes denoted by $SO^+(1,1)$), and that the mapping $\phi \to D(\phi)$ defines an isomorphism between this group and the group of real numbers under addition.

c) Show that there exists a matrix J (called the *generator* of the subgroup) such that every $D(\phi)$ can be written in the form

$$D(\phi) = e^{J\phi}$$

and determine J explicitly.

(6 points)

Solution

a) Let $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For D to be a Lorentz transformation, we must have

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2 - b^2 & ac - bd \\ ac - bd & c^2 - d^2 \end{pmatrix}$$

From this we obtain three constraints for the four numbers a, b, c, d:

(i) $a^2 - b^2 = 1$
(ii) $c^2 - d^2 = -1$
(iii) $ac - bd = 0$ 1pt

Now consider $\sinh\phi$, which maps $\mathbb{R}$ one-to-one onto itself. Therefore,
$\forall b \in \mathbb{R}\ \exists!\ \phi \in \mathbb{R}$ such that $b = \sinh\phi$
(i) $\Rightarrow a^2 = 1 + b^2 = 1 + \sinh^2\phi = \cosh^2\phi \Rightarrow a = \sigma \cosh\phi$, $\sigma = \pm 1$
Analogously, $c = \sinh\psi$
(ii) $\Rightarrow d^2 = 1 + c^2 = 1 + \sinh^2\psi = \cosh^2\psi \Rightarrow d = \tau \cosh\psi$, $\tau = \pm 1$ 1pt
Finally,
(iii) $\Rightarrow$

$$\begin{aligned} 0 &= \sigma \cosh\phi \sinh\psi - \tau \cosh\psi \sinh\phi \\ &= \cosh\phi \sinh\psi - \sigma\tau \cosh\psi \sinh\phi \\ &= \cosh(\sigma\tau\phi) \sinh\psi - \cosh\psi \sinh(\sigma\tau\phi) \\ &= \sinh(\psi - \sigma\tau\phi) \end{aligned}$$

$\Rightarrow \psi = \sigma\tau\phi$ 1pt

Therefore,

$$\begin{aligned} D_{\sigma,\tau}(\phi) = & \begin{pmatrix} \sigma\cosh\phi & \sinh\phi \\ \sinh(\sigma\tau\phi) & \tau\cosh(\sigma\tau\phi) \end{pmatrix} = \begin{pmatrix} \sigma\cosh\phi & \sinh\phi \\ \sigma\tau\sinh(\phi) & \tau\cosh(\phi) \end{pmatrix} \\ = & \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \begin{pmatrix} \sigma\cosh\phi & \sinh\phi \\ \sigma\sinh\phi & \cosh\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix} \\ = & \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} D(\phi) \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

with

$$D(\phi) = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}$$

where $\phi \in \mathbb{R}$ and $\sigma, \tau = \pm 1$, is the most general element of $O(1,1)$. 1pt

b) We have

(i)

$$\begin{pmatrix} \cosh\phi_1 & \sinh\phi_1 \\ \sinh\phi_1 & \cosh\phi_1 \end{pmatrix} \begin{pmatrix} \cosh\phi_2 & \sinh\phi_2 \\ \sinh\phi_2 & \cosh\phi_2 \end{pmatrix} = \begin{pmatrix} \cosh(\phi_1+\phi_2) & \sinh(\phi_1+\phi_2) \\ \sinh(\phi_1+\phi_2) & \cosh(\phi_1+\phi_2) \end{pmatrix}$$

$\Rightarrow D(\phi_1)D(\phi_2) = D(\phi_1+\phi_2) \Rightarrow$ closure ✓

(ii) Matrix multiplication is associative

(iii) $D(\phi = 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}_2$ is the neutral element

(iv) $D(-\phi) = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$

and $D(-\phi)D(\phi) = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow D(-\phi)$ is the inverse of $D(\phi)$

$\Rightarrow \{D(\phi)\}$ is a group SO$^+$(1,1) under matrix multiplication,

and (i), (iii), (iv) provide an isomorphism SO$^+$(1,1) $\cong$ $\mathbb{R}(+)$ 1pt

c) $e^{J\phi} = \mathbb{1}_2 + J\phi + \frac{1}{2} J^2\phi^2 + \ldots$

and $\cosh\phi = 1 + \frac{1}{2}\phi^2 + \frac{1}{4!}\phi^4 + \ldots$, $\sinh\phi = \phi + \frac{1}{3!}\phi^3 + \ldots$

Try $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad J^2 = \mathbb{1}_2$, $J^3 = J$ etc.

$\Rightarrow e^{J\phi} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \phi \\ \phi & 0 \end{pmatrix} + \begin{pmatrix} \phi^2/2 & 0 \\ 0 & \phi^2/2 \end{pmatrix} + \begin{pmatrix} 0 & \phi^3/3! \\ \phi^3/3! & 0 \end{pmatrix} + \ldots = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}$

$\Rightarrow J$ is the generator of SO$^+$(1,1) 1pt