**I.4.4. Symmetric tensors**

Let $V$ be an $n$-dimensional vector space over $K$ with some basis, let $f : V \times V \to K$ be a bilinear form, and let $t$ be the rank-2 tensor defined by $f$. Show that $f$ is symmetric, i.e. $f(x, y) = f(y, x)\ \forall x, y \in V$, if and only if the components of the tensor with respect to the given basis are symmetric, i.e., $t_{ij} = t_{ji}$.

(2 points)

**Solution**

Let $f(x, y)$ be symmetric, i.e., $f(x, y) = f(y, x) \quad \forall\ x, y \in V$.
The tensor components are defined by $t_{ij} = f(e_i, e_j)$

$$\Rightarrow \quad t_{ij} = f(e_i, e_j) = f(e_j, e_i) = t_{ji}$$

So the tensor is symmetric 1pt

Now let the tensor be symmetric, i.e., $t_{ij} = t_{ji}$. Then by definition of the tensor components we have

$$f(e_i, e_j) = f(e_j, e_i) \qquad \text{for all basis vectors } e_i, e_j$$

Then for any $x, y \in V$ we have

$$f(x, y) = x^i y^j f(e_i, e_j) = x^i y^j f(e_j, e_i) = y^j x^i f(e_j, e_i) = f(y, x)$$

So the form is symmetric. 1pt