I.4.1 Function space

Consider the set C of continuous functions $f : [0,1] \to \mathbb{R}$. Show that by suitably defining an addition on C, and a multiplication with real numbers, one can make C an additive vector space over $\mathbb{R}$.

(2 points)

Solution

On C, define

$$(f+g)(x) := f(x) + g(x)$$

If f and g are continuous, then so is the such defined $f+g$. $\Rightarrow$ Closure ✓

Furthermore, since $f(x) \in \mathbb{R}$, C inherits all other group properties of $(\mathbb{R}, +)$.

$\Rightarrow$ C is an additive group. 1pt

Now define multiplication with scalars $\lambda \in \mathbb{R}$ by

$$(\lambda f)(x) := \lambda\, f(x)$$

If f is continuous, then so is the such defined (λf).

Furthermore, since $\lambda \in \mathbb{R}$ and $f(x) \in \mathbb{R}$, this multiplication with scalars is bilinear and associative, as it inherits these properties from $\mathbb{R}$ under ordinary addition and multiplication of numbers.

Finally,

$$(1\, f)(x) = 1\, f(x) = f(x) \qquad \forall\, x \in [0,1] \qquad \Rightarrow \qquad 1\, f = f$$

$\Rightarrow$ C is an $\mathbb{R}$-vector space. 1pt