I.2.4 Subgroups

Let $(G, \vee)$ be a group and let $H \subset G$ with $H \neq \emptyset$. Show that H is a subgroup of G if and only if $a, b \in H$ implies $a \vee b^{-1} \in H$.

(5 points)

Solution

We first show that the condition is sufficient, i.e., that $a, b \in H \;\Rightarrow a \vee b^{-1} \in H$ implies that H is a subgroup. 1pt

Suppose $a, b \in H$. Then $a \vee b^{-1} \in H$ since H is a group.
In particular, $b = a \in H$ implies $a \vee a^{-1} = e \in H$
and if $a = e$, then $e \vee b^{-1} = b^{-1} \in H$ (*).
So the requirements **iii.** (existence of a neutral element) and **iv.** (existence of an inverse) from ch. I §2.1 Def. 1 are fulfilled. 1pt
Also, requirement **ii.** (associativity) is fulfilled since G and H share the associative operation $\vee$.

Now consider $a \vee b = a \vee (b^{-1})^{-1} \in H$, since from (*) we know that $b^{-1} \in H$ if $b \in H$.
So requirement **i.** (closure) is fulfilled.

Now we have shown that H is a group, i.e., the condition is sufficient. 1pt

Next we show that the condition is necessary, i.e., that $a, b \in H \;\not\Rightarrow a \vee b^{-1} \in H$ implies that H is not a subgroup. 1pt

Suppose $\exists a, b \in H$ such that $a \vee b^{-1} \notin H$.
In order for H to be a group, $b \in H$ must imply $b^{-1} \in H$.
Now we have $a, b^{-1} \in H$, but $a \vee b^{-1} \notin H$.
Thus requirement i (closure) is violated, and hence H is not a group. 1pt

Now we have shown that the condition is necessary, which completes the proof.