I.2.3 The group S_3

a) Compile the group table for the symmetric group S_3. Is S_3 abelian?

b) Find all subgroups of S_3. Which of these are abelian?

(7 points)

Solution

a) The elements of S_3 are

$$P_1 = \begin{pmatrix} 1\ 2\ 3 \\ 1\ 2\ 3 \end{pmatrix},\ P_2 = \begin{pmatrix} 1\ 2\ 3 \\ 1\ 3\ 2 \end{pmatrix},\ P_3 = \begin{pmatrix} 1\ 2\ 3 \\ 2\ 1\ 3 \end{pmatrix},\ P_4 = \begin{pmatrix} 1\ 2\ 3 \\ 3\ 1\ 2 \end{pmatrix},\ P_5 = \begin{pmatrix} 1\ 2\ 3 \\ 2\ 3\ 1 \end{pmatrix},\ P_6 = \begin{pmatrix} 1\ 2\ 3 \\ 3\ 2\ 1 \end{pmatrix}$$

1pt

With this representation, the group table reads

	P_1	P_2	P_3	P_4	P_5	P_6
P_1	P_1	P_2	P_3	P_4	P_5	P_6
P_2	P_2	P_1	P_5	P_6	P_3	P_4
P_3	P_3	P_4	P_1	P_2	P_6	P_5
P_4	P_4	P_3	P_6	P_5	P_1	P_2
P_5	P_5	P_6	P_2	P_1	P_4	P_3
P_6	P_6	P_5	P_4	P_3	P_2	P_1

S_3 is not abelian: E.g., $P_2 \circ P_3 = P_5$, but $P_3 \circ P_2 = P_4$. 1pt

b) Consider the group table from part a), and consider subsets of S_3 that contain

- 5 elements: $\{P_2, P_3, P_4, P_5, P_6\}$ does not contain $E = P_1$.
  $\{P_1, P_3, P_4, P_5, P_6\}$ is not closed since $P_3 \circ P_4 = P_2$.
  The other 4 candidates are not closed either. 1pt
- 4 elements: The subset must contain $P_1 \Rightarrow$ We need to conider only
  $\{P_1, P_2, P_3, P_4\}$ is not closed since $P_2 \circ P_3 = P_5$.
  $\{P_1, P_2, P_3, P_5\}$ is not closed since $P_3 \circ P_2 = P_4$.
  $\{P_1, P_2, P_4, P_5\}$ is not closed since $P_4 \circ P_2 = P_3$.
  $\{P_1, P_3, P_4, P_5\}$ is not closed since $P_3 \circ P_4 = P_2$.
  The other 6 candidates are not closed either. 1pt
- 3 elements: Consider $\{P_1, P_4, P_5\}$, whose group table is

	P_1	P_4	P_5
P_1	P_1	P_4	P_5
P_4	P_4	P_5	P_1
P_5	P_5	P_1	P_4

  This is an abelian subgroup!

  Whereas, $\{P_1, P_2, P_3\}$ is not closed since $P_2 \circ P_3 = P_5$,
  and the same for the other 8 candidates. 1pt
- 2 elements: $\{P_1, P_2\}$, $\{P_1, P_3\}$, and $\{P_1, P_6\}$ are abelian subgroups, whereas
  $\{P_1, P_4\}$ and $\{P_1, P_5\}$ are not closed. 1pt
- 1 element: $\{P_1\}$ trivially is an abelian subgroup, none of the other elements are. 1pt

Summary: The subgroups of S_3 are $\{P_1, P_4, P_5\}$, $\{P_1, P_2\}$, $\{P_1, P_3\}$, $\{P_1, P_6\}$, $\{P_1\}$.
They all are abelian.