CHAPTER 1: MARKOV CHAINS

THE GAMBLER'S RUIN PROBLEM:

CONSIDER A SEQUENCE OF RANDOM VARIABLES

$$X_0, X_1, \ldots$$

FORMULATED AS FOLLOWS.

YOU START WITH X_0 DOLLARS, AND FLIP A COIN WHICH COMES UP HEADS WITH PROBABILITY p.

IF THE COIN COMES UP HEADS, YOU WIN A DOLLAR, AND

$$X_1 = X_0 + 1.$$

IF IT COMES UP TALES, YOU LOSE A DOLLAR. THE GAME ENDS WHEN YOU RUN OUT OF MONEY.

THUS WE MAY HAVE A SITUATION LIKE

	H	T	H	H	T	T	T	H
10	11	10	11	12	11	10	9	10
X_0	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8

WHAT IS THE CONDITIONAL PROBABILITY

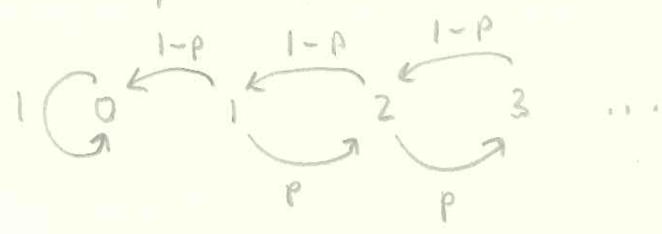
$$P\{X_8 = 10 \mid X_7 = 9, X_6 = 10, X_5 = 11, \ldots\} \quad ?$$
$$= P\{X_8 = 10 \mid X_7 = 9\} = p$$

MORE GENERALLY,

$$P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\}$$
$$= P\{X_{n+1} = j \mid X_n = i\}$$

THAT IS, ONLY THE INFORMATION FROM THE PREVIOUS R.V. IS NECESSARY TO COMPUTE THE PROBABILITY OF A RANDOM VARIABLE IN THE CHAIN.

WE MAY REPRESENT THE PICTORIALLY



LET US DEFINE THE TRANSITION PROBABILITY BY:

$$p(i,j) = P\{X_{n+1} = j \mid X_n = i\}$$

NOTICE IN THIS CASE THAT THE TRANSITION PROB. IS INDEPENDENT OF n (THAT IS, THE FAIRNESS OF THE COIN IS NOT CHANGING IN TIME). IN THIS SITUATION WE SAY WE ARE IN THE TEMPORALLY HOMOGENEOUS CASE.

NOTICE THAT

$$p(i,i+1) = 1-p \; ; \; p(i,i-1) = p \qquad 0 < i$$

$$p(0,0) = 1$$

FURTHER ASSUME THAT IF YOU EVER REACH N DOLLARS YOU "BREAK THE BANK" AND HAVE TO STOP PLAYING. THAT IS

$$p(N,N) = 1.$$

WE MAY CREATE AN $(N+1)\times(N+1)$ MATRIX FROM THIS INFORMATION

$$\left[\, p(i,j) \,\right]_{i,j=0}^{N}$$

FOR INSTANCE, WHEN $N=4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ p & 0 & 1-p & 0 & 0 \\ 0 & p & 0 & 1-p & 0 \\ 0 & 0 & p & 0 & 1-p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

THIS IS AN EXAMPLE OF A MARKOV CHAIN. MARKOV CHAINS ARE USUALLY DESCRIBED BY THEIR TRANSITION PROBABILITIES.

SUPPOSE WE ARE ON A MAGICAL PLANET WHERE THE WEATHER ON A PARTICULAR DAY DEPENDS (RANDOMLY) ONLY ON THE PREVIOUS DAYS WEATHER

SUPPOSE THE THREE WEATHER STATES ARE

1 = RAINY
2 = CLOUDY
3 = SUNNY

THE MARKOV CHAIN IS THEN COMPLETELY DETERMINED BY THE TRANSITION PROBABILITIES MATRIX, SAY

$$\begin{bmatrix} .4 & .6 & 0 \\ .2 & .5 & .3 \\ .1 & .7 & .2 \end{bmatrix} = [p(i,j)]$$

THUS

$P\{\text{cloudy} \to \text{sunny}\}$

$$p\{\text{SUNNY TODAY} \mid \text{CLOUDY YESTERDAY}\} = p(2,3) = \cancel{.7}\ \cancel{.7}\ .3$$

NOTICE THAT, WE MUST HAVE

$$\begin{aligned} &p\{\text{SUNNY TODAY} \mid \text{CLOUDY YESTERDAY}\} \\ +\ &p\{\text{CLOUDY TODAY} \mid \text{CLOUDY YESTERDAY}\} \\ +\ &p\{\text{RAINY TODAY} \mid \text{CLOUDY YESTERDAY}\} = 1 \end{aligned}$$

THAT IS

$$p(2,3) + p(2,2) + p(2,1) = 1$$

MORE GENERALLY WE MUST HAVE

$$\sum_j p(i,j) = 1.$$

$$p(i,j) = P\{i \to j\}$$

Q: WHAT IS THIS SAYING?

ANY MATRIX

$$[p(i,j)]_{i,j=1}^{N} \longleftarrow \text{or } 0$$

WITH

$$p(i,j) \geq 0 \quad \text{AND} \quad \sum_{j=1}^{N} p(i,j) = 1$$

SPECIFIES A MARKOV CHAIN!

MORE EXAMPLES AND THE TYPES OF QUESTIONS WE MAY BE INTERESTED IN

A MACHINE WITH 3 PARTS CAN FUNCTION AS LONG AS 2 PARTS ARE FUNCTIONAL. WHEN 2 PARTS ARE BROKEN, THE MACHINE IS FIXED BY THE NEXT DAY. IF WE DENOTE THE PARTS A, B, C AND THE STATE OF THE MACHINE BY ITS BROKEN PARTS (i.e ONE OF

$$\{0, A, B, C, AB, AC, BC\}$$

IF A, B, C BREAK INDEPENDENTLY WITH PROBABILITY .01, .02 AND .04 RESPECTIVELY

	0	A	B	C	AB	AC	BC
0	.93						
A	0						
B	0						
C	0						
AB	1	0	0	0	0	0	0
AC	1	0	0	0	0	0	0
BC	1	0	0	0	0	0	0

LET A^c, B^c, C^c BE THE EVENTS THAT PARTS A, B AND C DO NOT BREAK.

THEN, THE (0,0) ENTRY OF THE MATRIX IS GIVEN BY

$$P(A^c \cap B^c \cap C^c) = (.99)(.98)(.96) = .93$$

THE (A,0) ENTRY IS 0, SINCE IF A IS BROKEN AND NO OTHER PARTS ARE, THEN MACHINE IS NOT FIXED

EX 1: COMPLETE THE MATRIX

QUESTIONS WE MAY BE INTERESTED IN

Q: IF WE RUN THE MACHINE FOR, SAY N DAYS, HOW MANY OF EACH PART SHOULD WE EXPECT TO USE?

BRANCHING PROCESSES:

CONSIDER A POPULATION IN WHICH EACH INDIVIDUAL IN THE n^{TH} GENERATION INDEPENDENTLY GIVES BIRTH, PRODUCING k CHILDREN (WHO ARE IN GENERATION $n+1$) WITH PROBABILITY p_k.

LET X_n BE THE SIZE OF THE n^{TH} GENERATION. CLEARLY,

$$X_n \in \{0, 1, 2, \dots\}$$

IF WE LET $Y_1, Y_2, \dots$ BE INDEPENDENT RANDOM VARIABLES WITH

$$P\{Y_m = k\} = p_k$$

THEN WE MAY WRITE THE TRANSITION PROBABILITY AS

$$p(i,j) = P\{Y_1 + \dots + Y_i = j\}$$

CLEARLY $p(0,0) = 1$, AND WE SAY THAT 0 IS AN ABSORBING STATE.

Q: WHAT IS THE PROBABILITY THAT THE SPECIES DOES NOT DIE OUT?

THIS PROCESS IS CALLED THE GALTON-WATSON PROCESS

SUPPOSE WE HAVE N CHROMOSOMES, EACH OF WHICH HAS A GENE THAT CAN BE ONE OF TWO TYPES A OR a.

THE POPULATION OF CHROMOSOMES AT TIME $n+1$ IS FORMED FROM THAT AT TIME n BY SELECTING WITH REPLACEMENT (THUS A SINGLE CHROMOSOME COULD SHOW UP TWICE OR MORE IN THE NEXT GENERATION).

LET X_n DENOTE THE NUMBER OF A GENES AT TIME n. THEN X_n IS A MARKOV CHAIN WITH TRANSITION PROBABILITY

$$p(i,j) = \underbrace{\binom{N}{j}\left(\frac{i}{N}\right)^j\left(1-\frac{i}{N}\right)^{N-j}}_{\text{BINOMIAL DISTRIBUTION WITH PROB } \frac{i}{N}.}$$

Q: STARTING FROM $X_n = i$, WHAT IS THE PROBABILITY THAT THE POPULATES FIXATES IN THE ALL A STATE?

THIS IS CALLED THE WRIGHT-FISHER MODEL W/ NO MUTATIONS.

HW: 1.9.1 – 1.9.3

§1.2: MULTISTEP TRANSITION PROBABILITIES

$$p(i,j) = P(X_{n+1} = j \mid X_n = i)$$

LET US DEFINE

$$p^m(i,j) = P(X_{n+m} = j \mid X_n = i)$$

THAT IS, THIS IS THE PROBABILITY OF GOING FROM STATE i TO STATE j IN m STEPS.

FOR THERE WEATHER MARKOV CHAIN WE HAD TRANSITION MATRIX

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & .4 & .6 & 0 \\ 2 & .2 & .5 & .3 \\ 3 & .1 & .7 & .2 \end{array}$$

WHAT IS THE PROBABILITY THAT THE WEATHER WILL BE IN STATE 3 DAY AFTER NEXT GIVEN THAT IT IS IN STATE 1 TODAY.

THAT IS, FIND $p^2(1,3) = P\{X_2 = 3 \mid X_0 = 1\}$

$$\begin{aligned} p^2(1,3) = &\ P\{X_2 = 3, X_1 = 1 \mid X_0 = 1\} \\ &+ P\{X_2 = 3, X_1 = 2 \mid X_0 = 1\} \\ &+ P\{X_2 = 3, X_1 = 3 \mid X_0 = 1\} \end{aligned}$$

LET'S LOOK AT THE FIRST OF THESE:

$$P\{X_2=3, X_1=1 \mid X_0=1\} = \frac{P\{X_2=3, X_1=1, X_0=1\}}{P\{X_0=1\}}$$

$$= \frac{P\{X_2=3, X_1=1, X_0=1\}}{P\{X_1=1, X_0=1\}} \cdot \frac{P\{X_1=1, X_0=1\}}{P\{X_0=1\}}$$

$$= P\{X_2=3 \mid X_1=1, X_0=1\} \cdot P\{X_1=1 \mid X_0=1\}$$

$$= P\{X_2=3 \mid X_1=1\} \cdot P\{X_1=1 \mid X_0=1\}$$

$$= p(1,3) \cdot p(1,1)$$

A SIMILAR CALCULATION FOR THE OTHER TWO TERMS SHOWS THAT

$$p^2(1,3) = p(1,3) \cdot p(1,1) + p(2,3)\, p(1,2) + p(3,3) \cdot p(1,3)$$

$$= 0 + (.3)(.6) + (.2) \cdot 0 = .18$$

WE HAVE SHOWN THAT

$$P(X_2=j \mid X_0=i) = \sum_{k=1}^{3} p(i,k)\, p(k,j)$$

THAT IS $[p^2(i,j)] = [p(i,j)]^2$.

<u>THM</u>: THE m STEP TRANSITION PROBABILITY

$$P\{X_{n+m}=j \mid X_n=i\}$$

IS THE (i,j) ENTRY OF THE m^{TH} POWER OF THE TRANSITION MATRIX.

<u>THM</u>: (CHAPMAN-KOLMOGOROV EQUATION)

$$p^{m+n}(i,j) = \sum_k p^m(i,k)\, p^n(k,j).$$

NOTICE THAT THE CHAPMAN-KOLMOGOROV EQUATION IMPLIES THE FIRST THEOREM, SINCE

i) BY DEFINITION $p(X_{n+1}=j \mid X_n=i) = p(i,j)$

ii) IF WE ASSUME

$p(X_{n+m}=j \mid X_n=i)$ IS THE (i,j) ENTRY OF THE m^{TH} POWER OF THE TRANSITION MATRIX (THE INDUCTIVE HYPOTHESIS). THEN BY CHAPMAN KOLMOGOROV.

$$P(X_{n+m+1}=j \mid X_n=i)$$
$$= p^{m+1}(i,j) = \sum_k p^m(i,k)\,p(k,j)$$

WHICH IS THE (i,j) ENTRY OF THE $(m+1)$ TH POWER OF THE TRANSITION MATRIX.

Pf OF CHAPMAN-KOLMOGOROV:

$$P\{X_{m+n}=j \mid X_0=i\} = \sum_k P\{X_{m+n}=j, X_m=k \mid X_0=i\}$$

LOOKING AT THE SUMMAND,

$$P\{X_{m+n}=j, X_m=k \mid X_0=i\} = \frac{P\{X_{m+n}=j, X_m=k, X_0=i\}}{P\{X_0=i\}}$$

$$= \frac{P\{X_{m+n}=j, X_m=k, X_0=i\}}{P\{X_m=k, X_0=i\}} \cdot \frac{P\{X_m=k, X_0=i\}}{P\{X_0=i\}}$$

$$= P\{X_{m+n}=j \mid X_m=k, X_0=i\} \cdot P\{X_m=k \mid X_0=i\}$$
$$= P\{X_{m+n}=j \mid X_m=k\}\, P\{X_m=k \mid X_0=i\}$$
$$= p^n(k,j)\, p^m(i,k)$$

AND THE THEOREM FOLLOWS. ✓

HW: 1.9.4, 1.9.5

§ 1.3 :

DEF : GIVEN A STATE y OF A MARKOV CHAIN LET

$$T_y = \min\{n \geq 1 : X_n = y\}$$

BE THE TIME OF FIRST RETURN TO y.

(BEING THERE AT TIME 0 DOESN'T COUNT).

AND LET

$$g_y = P\{T_y < \infty \mid X_0 = y\}.$$

THAT IS g_y IS THE PROBABILITY OF RETURNING TO STATE y AT SOME POINT IN THE FUTURE, GIVEN THE CHAIN STARTED IN STATE y.

INTUITIVELY, WE EXPECT THE PROBABILITY THAT THE SYSTEM RETURNS TO y TWICE, GIVEN IT STARTED AT y IS g_y^2.

DEF : WE SAY T IS A STOPPING TIME IF THE OCCURANCE (OR NONOCCURANCE) OF THE EVENT "WE STOP AT TIME n," $\{T = n\}$ CAN BE DETERMINED BY LOOKING AT THE VALUES OF THE PROCESS UP TO THAT TIME.

NOTICE THAT T_y IS A STOPPING TIME SINCE

$$\{T_y = n\} = \{X_1 \neq y, X_2 \neq y, \ldots, X_{n-1} \neq y, X_n = y\}$$

STRONG MARKOV PROPERTY:

SUPPOSE T IS A STOPPING TIME. GIVEN THAT $T = n$ AND $X_T = y$, ANY OTHER INFORMATION ABOUT $X_0, \ldots, X_T$ IS IRRELEVANT FOR PREDICTING THE FUTURE, AND X_{T+k}, $k \geq 0$ BEHAVES LIKE THE MARKOV CHAIN WITH INITIAL STATE y.

TO VERIFY THIS WE NEED TO SHOW

$$P\{X_{T+1} = z \mid X_T = y, T = n\} = p(y, z)$$

Pf: LET $V_n = \{ (x_0, x_1, \ldots, x_n) :$ IF $X_0 = x_0, \ldots, X_n = x_n$ THEN $T = n$ AND $X_T = y \}$

THEN

$$P\{X_{T+1} = z, X_T = y, T = n\}$$
$$= \sum_{\vec{x} \in V_n} P\{X_{n+1} = z, X_n = x_n, \ldots, X_0 = x_0\}$$
$$= \sum_{\vec{x} \in V_n} P\{X_{n+1} = z \mid X_n = x_n, \ldots, X_0 = x_0\} \times P\{X_n = x_n, \ldots, X_0 = x_0\}$$
$$= \sum_{\vec{x} \in V_n} P\{X_{n+1} = z \mid X_n = y\} \times P\{X_n = x_n, \ldots, X_0 = x_0\}$$
$$= p(y,z) \sum_{\vec{x} \in V_n} P\{X_n = x_n, \ldots, X_0 = x_0\}$$
$$= p(y,z)\, P\{T = n, X_T = y\}$$

THUS

$$\frac{P\{X_{T+1} = z, X_T = y, T = n\}}{P\{T = n, X_T = y\}} = P\{X_{T+1} = z \mid X_T = y, T = n\}$$
$$= p(y,z). \checkmark$$

Q: WHERE DID WE USE THE FACT THAT T IS A STOPPING TIME?

Q: WHY ISN'T EVERY TIME A STOPPING TIME?

THE STRONG MARKOV PROPERTY FOR RETURN TIMES IMPLIES THAT THE PROBABILITY OF RETURNING TO STATE y GIVEN THAT WE HAVE RETURNED $(n-1)$ TIMES IS g_y.

THUS THE PROBABILITY OF RETURNING TWICE IS g_y^2, AND MORE GENERALLY THE PROBABILITY OF RETURNING n TIMES IS g_y^n.

DEF: IF $g_y < 1$ THEN THE PROBABILITY OF RETURNING n TIMES IS g_y^n

$$g_y^n \longrightarrow 0 \quad \text{AS} \quad n \longrightarrow \infty.$$

IN WHICH CASE, EVENTUALLY THE MARKOV CHAIN DOES NOT RETURN TO y, AND WE CALL y A TRANSIENT STATE.

IF ON THE OTHER HAND $g_y^n = 1$, THEN WE MAY EXPECT THE CHAIN TO RETURN TO y INFINITELY MANY TIMES (AT LEAST ASSUMING IT STARTS IN y), AND WE CALL y A RECURRENT STATE.

SUPPOSE WE HAVE THE TRANSITION MATRIX

	0	1	2	3	4	5
0	1	0	0	0	0	0
1	.6	0	.4	0	0	0
2	0	.6	0	.4	0	0
3	0	0	.6	0	.4	0
4	0	0	0	.6	0	.4
5	0	0	0	0	0	1

(GAMBLER'S RUIN.

CLEARLY STATES 0 AND 5 ARE RECURRENT. (ABSORBING STATES ARE STRONGLY RECURRENT)

HOW ABOUT THE STATE 1?

$$P(T_1 = \infty \mid X_0 = 1) \geq p(1,0) = .6 > 0$$

SO WITH POSITIVE PROBABILITY THE MARKOV CHAIN DOES NOT RETURN TO STATE 1. THAT IS

$$g_y < 1.$$

HOW ABOUT STATE 2?

NOTATION: $P_x\{A\} = P\{A \mid X_0 = x\}$

DEF: WE SAY STATE x COMMUNICATES WITH STATE y IF THERE IS A POSITIVE PROBABILITY OF REACHING y FROM x. THAT IS, IF

$$\rho_{xy} = P_x\{T_y < \infty\} > 0$$

IN THIS SITUATION WE WRITE $x \to y$.

NOTE THAT THE PROBABILITY OF GOING FROM x DIRECTLY TO y CAN BE 0, AND x CAN STILL COMMUNICATE WITH y. (i.e. IF x COMMUNICATES WITH y THERE MIGHT BE INTERMEDIATE STATES BETWEEN x AND y).

CLAIM: IF $x \to y$ AND $y \to z$ THEN $x \to z$

Pf: $P_x\{T_z < \infty\} \geq P_x\{T_y < \infty\} \cdot P_y\{T_z < \infty\} > 0.$

THM: IF $x \to y$ BUT $y \not\to x$ THEN x IS TRANSIENT

THIS IS OBVIOUS RIGHT? WHAT IS THE INTUITIVE REASON?

Pf: LET $K = \min\{k : p^k(x,y) > 0\}$

(THIS IS THE SMALLEST NUMBER OF STEPS WE MAY TAKE TO GET FROM x TO y).

SINCE $p^K(x,y) > 0$ THERE MUST BE SOME SEQUENCE $y_1, y_2, \ldots, y_{K-1}$ SUCH THAT

$$p(x,y_1)\,p(y_1,y_2)\cdots p(y_{K-1},y) > 0$$

SINCE K IS MINIMAL, ALL THE $y_i \neq y$, AND

$$P_x(T_x = \infty) \geq p(x,y_1)\,p(y_1,y_2)\cdots p(y_{K-1},y)\,(1-\rho_{yx})$$

WHY?

BUT, IF x IS RECURRENT THEN THE RIGHT-HAND SIDE MUST BE 0 WHICH WOULD IMPLY

$$\rho_{yx} = 1 \quad ⨳ \quad \text{SINCE } \rho_{yx} = 0.$$

✓

RECALL THE WEATHER CHAIN

$$\begin{bmatrix} .4 & .6 & 0 \\ .2 & .5 & .3 \\ .1 & .7 & .2 \end{bmatrix}$$

TO SHOW 1 IS A RECURRENT STATE

FIRST NOTICE THAT

$$P\{X_{n+1} = 1\} \geq \min\{ p(1,1),\ p(2,1),\ p(3,1)$$

$$\geq .1$$

THUS $P_1\{T_1 > n\} \leq (.9)^n \longrightarrow 0$ AS $n \to \infty$

HENCE, $P_1\{T_1 < \infty\} = 1$.

THIS ARGUMENT WORKS FOR STATE 2 AS WELL.

FOR STATE 3, WE CANNOT USE THIS ARGUMENT, HOWEVER

$$p^2 = \begin{bmatrix} .28 & .54 & .18 \\ .21 & .58 & .21 \\ .20 & .55 & .25 \end{bmatrix}$$

HENCE

$$P\{X_{n+2} = 3 \mid X_n = x\} \geq .18$$

AND HENCE

$$P_3\{T_3 > 2k\} \leq (.82)^k \longrightarrow 0 \quad \text{AS} \quad k \to \infty$$

AND HENCE 3 IS RECURRENT.

THE PEDESTRIAN LEMMA:

SUPPOSE $P_x(T_y \leq k) \geq \alpha > 0$ FOR ALL STATES x.
THEN

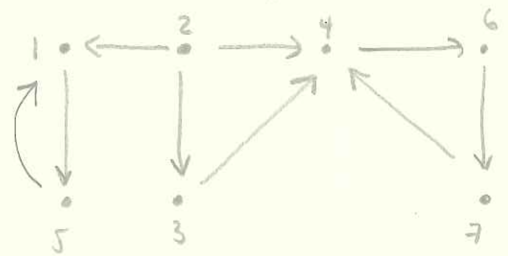
$$P_x(T_y > nk) \leq (1-\alpha)^n.$$

THIS FOLLOWS FROM THE OBSERVATION THAT IF A PEDESTRIAN HAS A PROBABILITY OF AT LEAST α OF BEING KILLED WHILE CROSSING THE STREET, THEN THE PROBABILITY OF CROSSING THE STREET SUCCESSFULLY n TIMES IS AT MOST $(1-\alpha)^n$.

A SEVEN-STATE CHAIN

$$\begin{bmatrix} .3 & 0 & 0 & 0 & .7 & 0 & 0 \\ .1 & .2 & .3 & .4 & 0 & 0 & 0 \\ 0 & 0 & .5 & .5 & 0 & 0 & 0 \\ 0 & 0 & 0 & .5 & 0 & .5 & 0 \\ .6 & 0 & 0 & 0 & .4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .2 & .8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

LET US MAKE A (DIRECTED) GRAPH WHICH CONNECTS THE INTEGERS i TO j IF $p(i,j) > 0$ AND $i \neq j$.



THIS IMPLIES THAT 2 COMMUNICATES WITH 1 BUT 1 DOES NOT COMMUNICATE WITH 2.

$\Rightarrow$ 2 IS TRANSIENT

LIKEWISE 3 IS TRANSIENT.

THE REMAINING STATES SEPERATE INTO TWO MUTUALLY COMMUNICATING SETS $\{1,5\}$ AND $\{4,6,7\}$

WE WILL SHOW THAT THESE SETS ARE RECURRENT.

<u>DEF</u>: A SET A OF STATES IS CLOSED IF $i \in A$ AND $j \in A^c$ IMPLIES THAT $p(i,j) = 0$.

THAT IS, A SET A IS CLOSED IF, ONCE YOU GET IN YOU CAN'T GET OUT.

FOR THE PREVIOUS EXAMPLE

$\{1,5\}$ AND $\{4,6,7\}$ ARE CLOSED.

SO ARE $\{1,4,5,6,7\}$, $\{1,2,3,4,5,6,7\}$.

DEF: A SET B IS IRREDUCIBLE IF WHENEVER $i,j \in B$ i COMMUNICATES WITH j

$\{1,5\}$ AND $\{4,6,7\}$ ARE IRREDUCIBLE

$\{1,4,5,6,7\}$ IS NOT IRREDUCIBLE SINCE $1 \not\to 4$.

THM: IF C IS A FINITE CLOSED AND IRREDUCIBLE SET, THEN ALL STATES IN C ARE RECURRENT.

THM: IF THE SPACE OF STATES S IS FINITE, THEN S CAN BE WRITTEN AS THE DISJOINT UNION

$$S = T \cup R_1 \cup \cdots \cup R_k$$

WHERE T IS A SET OF TRANSIENT STATES AND $R_1, \dots, R_k$ ARE CLOSED IRREDUCIBLE SETS OF RECURRENT STATES.

Pf: DEFINE

$$T = \{x : \exists y \ni x \to y \text{ AND } y \not\to x\}$$

BY A PREVIOUS THEOREM ALL STATES IN T ARE TRANSIENT.

NEXT WE WILL SHOW THAT THE REMAINING STATES ARE RECURRENT.

CHOOSE $x \in S \setminus T$ AND LET

$$C_x = \{y : x \to y\}$$

SINCE $x \notin T$, IF $x \to y$ THEN $y \to x$.

TO SEE THAT C_x IS CLOSED SUPPOSE $z \notin C_x$ THEN

$$x \not\to z, \text{ AND THUS } p(x,z) = 0.$$

TO CHECK IRREDUCIBILITY, NOTICE THAT IF $y, z \in C_x$ THEN,

$$x \to y \quad \text{AND} \quad x \to z$$

SINCE $x \notin T$ WE HAVE $y \to x$

THUS $y \to x \to z$ AND THEREFORE $y \to z$.

THAT IS C_x IS IRREDUCIBLE.

SO, LET $R_1 = C_x$ IF $S = T \cup R_1$ WE'RE DONE.

OTHERWISE, CHOOSE $y \in S \setminus (T \cup R_1)$ AND

CONSTRUCT $R_2 = C_y$. CONTINUE IN THIS MANNER UNTIL

$$S = T \cup R_1 \cup \ldots \cup R_k. \checkmark$$

IT REMAINS TO PROVE THE THEOREM BEFORE THIS. WE WILL DO THIS THROUGH A SEQUENCE OF LEMMAS

LEMMA: IF x IS RECURRENT AND $x \to y$ THEN y IS RECURRENT.

WHY? EVERY TIME THE CHAIN RETURNS TO x, IT HAS A POSITIVE PROBABILITY OF HITTING y BEFORE RETURNING TO x, THUS IT WILL NOT RETURN TO x INFINITELY MANY TIMES WITHOUT HITTING y AT LEAST ONCE.

LEMMA: IN A FINITE CLOSED SET THERE HAS TO BE AT LEAST ONE RECURRENT STATE.

WHY? PIGEON-HOLE PRINCIPAL.

WE WILL NOT PROVIDE MORE DETAILS THAN THIS.

HW: 1.9.6, 1.9.7, 1.9.9

§ 1.4: LIMIT BEHAVIOR

IF y IS TRANSIENT THEN $\forall x$

$$p^n(x,y) \longrightarrow 0$$

WHAT HAPPENS WITH $p^n(x,y)$ IF y IS RECURRENT?

THIS OBVIOUSLY DEPENDS ON THE MARKOV CHAIN, BUT IN MOST CASES $p^n(x,y)$ CONVERGES TO A POSITIVE LIMIT.

EHRENFEST CHAIN

SUPPOSE WE HAVE TWO URNS WITH A TOTAL OF N BALLS BETWEEN THEM. LET X_n DENOTE THE NUMBER OF BALLS IN THE LEFT URN AT TIME n.

X_{n+1} IS GIVEN FROM X_n BY CHOOSING ONE OF THE N BALLS AT RANDOM AND MOVING IT TO THE OTHER URN.

THE TRANSITION MATRIX WHEN $N=3$ IS

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{bmatrix} = P$$

IT CAN BE COMPUTED

$$\begin{bmatrix} 1/3 & 0 & 2/3 & 0 \\ 0 & 7/9 & 0 & 2/9 \\ 2/9 & 0 & 7/9 & 0 \\ 0 & 2/3 & 0 & 1/3 \end{bmatrix}$$

AND THE PATTERN OF ZEROS ALTERNATES WITH n. THUS, FOR INSTANCE

$$p^n(m,m) = 0 \quad \text{IF } n \text{ IS ODD} \ldots$$

DEF : THE PERIOD OF A STATE IS THE LARGEST NUMBER THAT WILL DIVIDE ALL THE n FOR WHICH

$$p^n(x,x) > 0$$

THAT IS

$$\mathrm{PER}(x) = \gcd\{ n \geq 1 : p^n(x,x) > 0 \}$$

IN THE PREVIOUS EXAMPLE

$$\mathrm{PER}(x) = 2$$

FOR ALL STATES x.

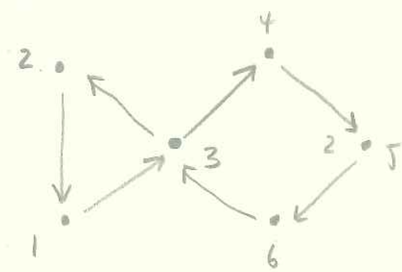
EX: 1.4.1

LEMMA: IF $p(x,x) > 0$ THEN x HAS PERIOD 1.

THIS IS CLEAR FROM THE DEFINITION.

$p(x,x) > 0$ IS A SUFFICIENT CONDITION, TO HAVE PERIOD 1, BUT IT IS NOT A NECESSARY ONE.

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$



$I_1 = \{3, 7, \dots\}$

$I_2 = \{3, 7, \dots\}$

$I_3 = \{3, 4, \dots\}$

$I_4 = \{4, 7, \dots\}$

$I_5 = \{4, 7, \dots\}$

$I_6 = \{4, 7, \dots\}$

$\Rightarrow \mathrm{PER}(x) = \gcd I_x = 1$

FOR ALL STATES x.

LEMMA: IF x HAS PERIOD 1 THEN $\exists\, n_0 \ni$ IF $n > n_0$

$$p^n(x,x) > 0$$

THAT IS $n \in I_x$.

(THIS PROOF RELIES ON NUMBER THEORY, SO WE WILL NOT REPORT IT).

LEMMA: IF $x \longrightarrow y$ AND $y \longrightarrow x$ THEN x AND y HAVE THE SAME PERIOD.

Pf: SUPPOSE $\mathrm{PER}(x) = c > d = \mathrm{PER}(y)$

LET k AND m BE SUCH THAT

$$p^k(x,y) > 0 \quad \text{AND} \quad p^m(y,x) > 0$$

NOTICE THAT

$$p^{k+m}(x,x) \geq p^k(x,y)\, p^m(y,x) > 0$$

THUS $k+m \in I_x$. NOW $c \mid n$ FOR ALL $n \in I_x$.

$\Rightarrow \quad c \mid k+m$ (EXPLAIN DIVIDES NOTATION)

NOW, LET ℓ BE AN INTEGER WITH $p^\ell(y,y) > 0$ THEN

$$p^{k+\ell+m}(x,x) \geq p^k(x,y)\, p^\ell(y,y)\, p^m(y,x) > 0$$

$\Rightarrow \quad k+\ell+m \in I_x$ AND

$$c \mid k+\ell+m$$

NOW, $c \mid k+m$ AND $c \mid k+\ell+m$

$$\Rightarrow c \mid \ell$$

BUT THEN $c \mid \ell$ FOR ALL $\ell \in I_y$

THIS IS A CONTRADICTION SINCE WE HAVE FOUND A COMMON DIVISOR GREATER THAN k. ✓

SYMMETRIC REFLECTING RANDOM WALK ON THE LINE

SUPPOSE WE HAVE A STATE SPACE $\{0, 1, \ldots, N\}$ AND AT EACH STEP THE SYSTEM HAS A 50/50 CHANCE OF STEPPING LEFT OR STEPPING RIGHT BY ONE STEP, WITH THE EXCEPTION THAT IF THE SYSTEM IS AT 0 IT HAS A 50/50 CHANCE OF STEPPING RIGHT OR STAYING PUT AND IF THE SYSTEM IS AT STATE N IT HAS A 50/50 CHANCE OF STEPPING LEFT OR STAYING PUT.

FOR INSTANCE IF $N = 4$

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 2 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 3 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 4 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}$$

NOTICE THAT $\forall x \in S$ WE HAVE $x \to 1$.

MOREOVER, $p(1,1) > 0 \Rightarrow \text{PER}(1) = 1$

THUS $\text{PER}(x) = 1 \quad \forall x \in S$.

IN INVESTIGATING THE LIMITING BEHAVIOR OF A MARKOV CHAIN WE HAVE SO FAR INVESTIGATED THE PERIODICITY OF STATES.

WE ARE INTERESTED ALSO IN THE LONG-TERM PROBABILITY OF OUR CHAIN BEING IN A PARTICULAR STATE, THAT IS, WE ARE INTERESTED IN PROBABILITIES OF THE FORM

$$\lim_{n \to \infty} p^n(x,y) \quad \text{(ASSUMING THIS LIMIT EXISTS).}$$

__DEF__: LET $P = [p(x,y)]$. A __STATIONARY DISTRIBUTION__ IS A VECTOR

$$\vec{\pi} = (\pi(x))_{x \in S} \quad \text{(ROW VECTOR)}$$

WITH

$$\sum_y \pi(y) = 1, \quad \text{AND} \quad \vec{\pi} P = \vec{\pi}.$$

THAT IS, $\vec{\pi}$ IS A STATIONARY DISTRIBUTION IF $\vec{\pi}$ IS A VECTOR OF PROBABILITIES WHICH IS A (LEFT) EIGENVECTOR OF P.

IN OTHER NOTATION, THE CONDITION $\vec{\pi} P = \vec{\pi}$ CAN BE WRITTEN

$$\sum_x \pi(x)\, p(x,y) = \pi(y)$$

THE NAME "STATIONARY DISTRIBUTION" COMES FROM THE FOLLOWING PROPERTY

LEMMA: IF X_0 HAS DISTRIBUTION π, i.e.

$$P\{X_0 = y\} = \pi(y)$$

(AND π IS A STATIONARY DISTRIBUTION) THEN SO DOES X_n FOR ALL $n \geq 1$.

Pf: $$P\{X_n = y\} = \underbrace{\vec{\pi}\, P^N}\,(y)$$

NOTICE THIS IS A VECTOR, THUS $\vec{\pi} P^N(y)$ IS THE COORDINATE OF THIS VECTOR CORRESPONDING TO STATE y.

BUT $$\vec{\pi} P^n = (\vec{\pi} P) P^{n-1} = \vec{\pi} P^{n-1}$$
$$= \dots = \pi.$$

DEF: WE SAY A MARKOV CHAIN (X_n), OR A TRANSITION MATRIX P IS IRREDUCIBLE IF THE SET OF STATES IS IRREDUCIBLE

A MARKOV CHAIN (OR TRANSITION MATRIX) IS SAID TO BE APERIODIC IF $\mathrm{PER}(x) = 1$ FOR ALL STATES x.

NOTICE THAT IF (X_n) IS IRREDUCIBLE AND $\mathrm{PER}(x) = 1$ FOR SOME STATE x, THEN $\mathrm{PER}(y) = 1$ FOR ALL STATES y.

WHY?

THM: (CONVERGENCE THEOREM)

SUPPOSE P IS IRREDUCIBLE, APERIODIC AND HAS STATIONARY DISTRIBUTION π. THEN, AS $n \to \infty$

$$P^n(x,y) \longrightarrow \pi(y).$$

TO IDENTIFY THE STATIONARY DISTRIBUTION IT SUFFICES TO FIND THE VALUE OF c, FOR WHICH

$$\pi_1 + \pi_2 + \pi_3 = \frac{19}{18}c + \frac{8}{3}c + c = 1$$

THAT IS,

$$c\left(\frac{19}{18} + \frac{8}{3} + 1\right) = c\left(\frac{19}{18} + \frac{48}{18} + \frac{18}{18}\right) = \frac{85}{18}c = 1$$

THUS

$$\vec{\pi} = \left(\frac{19}{85}, \frac{48}{85}, \frac{18}{85}\right)$$ IS THE STATIONARY DISTRIBUTION.

ONE CAN CHECK THE CONVERGENCE THEOREM IN THIS CASE BY SEEING, FOR INSTANCE

$$p^8 = \begin{bmatrix} .22355 & .56470 & .21175 \\ .22352 & .56471 & .21177 \\ .22352 & .56471 & .21177 \end{bmatrix}$$

AND $.2235 \approx \frac{19}{85}$, $.5647 \approx \frac{48}{85}$, $.2118 \approx \frac{18}{85}$

EXISTENCE AND UNIQUENESS OF STATIONARY DISTRIBUTIONS

THE CONVERGENCE THEOREM IMMEDIATELY IMPLIES THE FOLLOWING COROLLARY:

COR: IF p IS IRREDUCIBLE THEN THE STATIONARY DISTRIBUTION (IF IT EXISTS) IS UNIQUE.

Pf: SUPPOSE p IS APERIODIC, AND $\vec{\pi}_1$ AND $\vec{\pi}_2$ ARE TWO STATIONARY DISTRIBUTIONS, THEN

$$\pi_1(y) = \lim_{n\to\infty} p^n(x,y) = \pi_2(y)$$

TO REMOVE THE APERIODIC CONDITION LET I BE THE TRANSITION PROBABILITY FOR THE CHAIN THAT NEVER MOVES, THAT IS

$$I(x,x) = 1 \quad \text{FOR ALL STATES } x.$$

FOR THE WEATHER CHAIN WE HAVE

$$P = \begin{bmatrix} .4 & .6 & 0 \\ .2 & .5 & .3 \\ .1 & .7 & .2 \end{bmatrix}$$

SUPPOSE WE HAVE A VECTOR $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ SUCH THAT

$\vec{\pi} P = \vec{\pi}$ THEN (BY TAKING TRANSPOSES)

$$\begin{bmatrix} .4 & .2 & .1 \\ .6 & .5 & .7 \\ 0 & .3 & .2 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix}$$

(WE WRITE THIS IN THIS FORM, SINCE TYPICALLY WE USE COLUMN VECTORS IN LINEAR ALGEBRA).

IF THERE IS A STATIONARY DISTRIBUTION, THEN 1 IS AN EIGENVALUE AND

$$P - I = \begin{bmatrix} -.6 & .2 & .1 \\ .6 & -.5 & .7 \\ 0 & .3 & -.8 \end{bmatrix} \quad \text{IS SINGULAR}$$

NOW

$$\sim \begin{bmatrix} .6 & -.2 & -.1 \\ 0 & -.3 & .8 \\ 0 & .3 & -.8 \end{bmatrix} \sim \begin{bmatrix} .6 & -.2 & -.1 \\ 0 & -.3 & .8 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1/3 & -1/6 \\ 0 & 1 & -8/3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/6 - 8/9 \\ 0 & 1 & -8/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{-1}{6} - \frac{8}{9} = \frac{-3}{18} - \frac{16}{18} = \frac{-19}{18}$$

THUS, THE NULL SPACE OF $P - I$ IS SPANNED BY

$$\left\{ \begin{bmatrix} 19/18 \\ 8/3 \\ 1 \end{bmatrix} \right\}, \text{ THAT IS}$$

$$\text{NULL}(P - I) = \left\{ \begin{bmatrix} 19/18\, c \\ 8/3\, c \\ c \end{bmatrix} : c \in \mathbb{R} \right\}$$

DEFINE A NEW TRANSITION MATRIX $\hat{P}$ BY

$$\hat{P} = \frac{I + P}{2}$$

THAT IS, $\hat{P}$ IS THE CHAIN GIVEN BY EITHER DOING NOTHING OR TAKING A STEP ACCORDING TO P.
WITH PROBABILITY $\frac{1}{2}$

SINCE $\hat{P}(x, x) \geq \frac{1}{2}$ FOR ALL STATES x, WE HAVE THAT $\hat{P}$ IS APERIODIC.

NOW, IF $\vec{\pi}$ IS A STATIONARY DISTRIBUTION FOR P, THEN

$$\vec{\pi}\hat{P} = \vec{\pi}\left(\frac{I + P}{2}\right) = \vec{\pi}\frac{I}{2} + \vec{\pi}\frac{P}{2}$$

$$= \frac{\vec{\pi}}{2} + \frac{\vec{\pi}}{2} = \vec{\pi}$$

THAT IS, IF $\vec{\pi}$ IS A STATIONARY DISTRIBUTION FOR P, THEN $\vec{\pi}$ IS A STATIONARY DISTRIBUTION FOR $\hat{P}$.

SINCE $\hat{P}$ IS APERIODIC, IT CAN HAVE ONLY ONE STATIONARY DISTRIBUTION. THUS P HAS AT MOST ONE STATIONARY DISTRIBUTION. ✓

<u>THM</u>: IF THE STATE SPACE IS FINITE THEN THERE IS AT LEAST ONE STATIONARY DISTRIBUTION.

<u>THM</u>: (STRONG LAW FOR MARKOV CHAINS).

SUPPOSE P IS IRREDUCIBLE AND HAS STATIONARY DISTRIBUTION π. LET $r(x)$ BE THE REWARD WE EARN IN STATE x, AND SUPPOSE

$$\sum \pi(x)\,|r(x)| < \infty.$$

THEN AS $n \to \infty$,

$$\frac{1}{n}\sum_{k=1}^{n} r(X_k) \longrightarrow \sum_{x} \pi(x)\, r(x).$$

THE NAME "STRONG LAW" FOLLOWS FROM THE ANALOGY WITH THE STRONG LAW OF LARGE NUMBERS:

IF $X_1, X_2, \ldots$ ARE INDEPENDENT RANDOM VARIABLES WITH DENSITY $\pi(x)$, THEN THE STRONG LAW OF LARGE NUMBERS APPLIED TO $r(X_1), r(X_2), \ldots$ SAYS THAT

$$\frac{1}{n}\sum_{k=1}^{n} r(X_k) \longrightarrow E[r(X)] = \sum_{x} \pi(x) r(x)$$

OF COURSE, IN OUR SITUATION $X_1, X_2, \ldots$ ARE NOT INDEPENDENT (NOR IN GENERAL DO THEY ALL HAVE DENSITY $\pi(x)$!), BUT NONETHELESS WE CAN DRAW THE SAME CONCLUSION.

IN THE CASE WHERE

$$r(y) = \begin{cases} 1 & \text{IF } y = x \\ 0 & \underline{\qquad\qquad} \end{cases} \qquad (*)$$

THEN

$$\frac{1}{n}\sum_{k=1}^{n} r(X_k) = \text{PROPORTION OF TIME THE CHAIN SPENDS IN STATE } x \text{ UP THROUGH TIME STEP } n.$$

$$\longrightarrow \pi(x) \quad \text{AS} \quad n \longrightarrow \infty.$$

THUS $\pi(x)$ GIVES THE ASYMPTOTIC PROPORTION OF TIME SPENT IN STATE x.

COR: LET $N_n(y) = \sum_{k=1}^{n} r(X_k)$ WHERE r IS AS IN $(*)$

THEN

$$\lim_{n\to\infty} \frac{N_n(y)}{n} = \pi(y).$$

SUPPOSE WE HAVE A REPAIR CHAIN WITH TRANSITION MATRIX

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 12 \\ 13 \\ 23 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 12 & 13 & 23 \end{array} \\
\begin{bmatrix}
.93 & .01 & .02 & .04 & 0 & 0 & 0 \\
0 & .94 & 0 & 0 & .02 & .04 & 0 \\
0 & 0 & .95 & 0 & .01 & 0 & .04 \\
0 & 0 & 0 & .97 & 0 & .01 & .02 \\
1 & 0 & & & & & \\
1 & & & & & & \\
1 & & & & & &
\end{bmatrix}
\end{array} = P$$

IF WE RUN THE MACHINE FOR 1800 DAYS, HOW MANY OF EACH PART SHOULD WE EXPECT TO REPLACE?

NOTICE THAT THE EQUATION

$$\vec{\pi} P = \vec{\pi}$$

IMPLIES (SECOND COLUMN)

$$.01\,\pi(0) + .94\,\pi(1) = \pi(1)$$

AND SETTING $\pi(0) = c \Rightarrow$

$$.01\,c = .06\,\pi(1) \Rightarrow \pi(1) = \frac{c}{6}$$

THE THIRD COLUMN GIVES

$$.02\,c + .95\,\pi(2) = \pi(2)$$

$$\Rightarrow \pi(2) = \frac{2c}{5}$$

THE FOURTH COLUMN GIVES

$$.04\,c + .97\,\pi(3) = \pi(3)$$

$$\Rightarrow \pi(3) = \frac{4c}{3}$$

CONTINUING IN THIS MANNER, WE FIND

$$\pi(12) = \frac{22c}{3000}, \quad \pi(13) = \frac{60c}{3000}, \quad \pi(23) = \frac{128c}{3000}$$

SUMMING THE π'S GIVES

$$c + \frac{c}{6} + \frac{2c}{5} + \frac{4c}{3} + \frac{22c}{3000} + \frac{60c}{3000} + \frac{128c}{3000} = \frac{8910c}{3000}$$

AND THUS

$c = \frac{3000}{8910}$ (WHY?). THAT IS

$$\pi(12) = \frac{22}{8910}, \quad \pi(13) = \frac{60}{8910} \text{ AND } \pi(23) = \frac{128}{8910}$$

WE EXPECT TO NEED A NEW PART 1

$\frac{22+60}{8910}$ OF THE TIME

AND THUS WE EXPECT TO NEED

$\frac{82}{8910} \cdot 1800 = 16.56$ OF THIS PART

SIMILARLY WE NEED

$\frac{128+22}{8910} \cdot 1800 = 30.3$ OF PART 2 AND

$\frac{188}{8910} \cdot 1800 = 37.98$ OF PART 3.

HW: 1.9.11, 1.9.12, 1.9.14, 1.9.15, 1.9.17

§ 1.5: SOME SPECIAL EXAMPLES

DEF: A TRANSITION MATRIX P IS SAID TO BE DOUBLY STOCHASTIC IF ITS COLUMNS SUM TO 1.

$$\sum_x p(x,y) = 1$$

(CLEARLY, $\sum_y p(x,y) = 1$ ALSO).

THM: IF P IS DOUBLY STOCHASTIC FOR A MARKOV CHAIN WITH N STATES, THEN THE UNIFORM DISTRIBUTION

$\pi(x) = \frac{1}{N} \quad \forall x$ IS A STATIONARY DISTRIBUTION.

Pf:

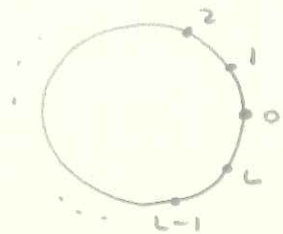
$$\sum_x \pi(x) p(x,y) = \frac{1}{N} \sum_x p(x,y) = \frac{1}{N}$$

THUS $\vec{\pi} P = \vec{\pi}$.

IT IS CLEAR FROM THE PROOF THAT IF $(1/N, 1/N, \ldots, 1/N)$ IS A STATIONARY DISTRIBUTION, THEN P IS DOUBLY STOCHASTIC.

NEAREST NEIGHBOR RANDOM WALK ON THE UNIT CIRCLE.

SUPPOSE THE STATE SPACE IS $\{0, 1, 2, \ldots, L\}$



AT EACH STEP THE RANDOM WALKER GOES LEFT (COUNTER CLOCKWISE) WITH PROBABILITY a, AND RIGHT (CLOCKWISE) WITH PROBABILITY $1-a$.

WHEN $L=4$, THE TRANSITION MATRIX IS:

$$\begin{bmatrix} 0 & a & 0 & 0 & 1-a \\ 1-a & 0 & a & 0 & 0 \\ 0 & 1-a & 0 & a & 0 \\ 0 & 0 & 1-a & 0 & a \\ a & 0 & 0 & 1-a & 0 \end{bmatrix} = P$$

IT IS EASY TO VERIFY THAT P IS DOUBLY STOCHASTIC AND THUS THE STATIONARY DISTRIBUTION IS

$$\vec{\pi} = (1/5, 1/5, 1/5, 1/5, 1/5).$$

EX 1.5.2

BIRTH AND DEATH CHAINS:

A BIRTH AND DEATH CHAIN IS A MARKOV CHAIN WITH STATE SPACE

$$S = \{\ell, \ell+1, \ldots, r-1, r\}$$

WITH THE PROPERTY THAT $p(x,y) = 0$ WHEN $|x-y| > 1$.

THESE CHAINS ARE (RELATIVELY) EASY TO STUDY SINCE THEIR STATIONARY DISTRIBUTION (IF IT EXISTS) SATISFIES THE FOLLOWING

DETAILED BALANCE CONDITION:

$$\pi(x)\, p(x,y) = \pi(y)\, p(y,x)$$

Pf:

$$\pi(y) = \sum_x \pi(x)\, p(x,y)$$

$$= \pi(y-1)\,p(y-1,y) + \pi(y)\,p(y,y) + \pi(y+1)\,p(y+1,y)$$

AND THUS

$$\pi(y)\big(1-p(y,y)\big) = \pi(y-1)\,p(y-1,y) + \pi(y+1)\,p(y+1,y)$$

NOW,

$$1-p(y,y) = p(y,y-1) + p(y,y+1)$$

AND HENCE,

$$\pi(y)\,p(y,y-1) + \pi(y)\,p(y,y+1)$$
$$= \pi(y-1)\,p(y-1,y) + \pi(y+1)\,p(y+1,y)$$

NOTICE THAT IF $y = \ell$ THIS GIVES

$$\pi(\ell)\,p(\ell,\ell+1) = \pi(\ell+1)\,p(\ell+1,\ell)$$

AND, IF $y = \ell+1$

$$\pi(\ell+1)\,p(\ell+1,\ell) + \pi(\ell+1)\,p(\ell+1,\ell+2)$$
$$= \pi(\ell)\,p(\ell,\ell+1) + \pi(\ell+2)\,p(\ell+2,\ell+1)$$

$$\Rightarrow \pi(\ell+1)\,p(\ell+1,\ell+2) = \pi(\ell+2)\,p(\ell+2,\ell+1)$$

CONTINUING IN THIS MANNER, WE HAVE

$$\pi(x)\,p(x,x+1) = \pi(x+1)\,p(x+1,x). \quad \forall x \in S$$

EXERCIZE:

SHOW THAT

$$\pi(x)\,p(x,x-1) = \pi(x-1)\,p(x-1,x).$$

✓

THE EHRENFEST CHAIN.

SUPPOSE
$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 1 & 1/3 & 0 & 2/3 & 0 \\ 2 & 0 & 2/3 & 0 & 1/3 \\ 3 & 0 & 0 & 1 & 0 \end{array} = P$$

(THE $n=3$ CASE)

THE DETAILED BALANCE EQUATIONS SAY

$$\pi(0)p(0,1) = \pi(1)p(1,0) \Rightarrow \pi(0) = \pi(1)/3$$

$$\pi(1)p(1,2) = \pi(2)p(2,1) \Rightarrow 2\pi(1)/3 = 2\pi(2)/3$$

$$\pi(2)p(2,3) = \pi(3)p(3,2) \Rightarrow \pi(2)/3 = \pi(3)$$

NOW, IF $\pi(0) = c$ THEN

$$\pi(0) = c, \quad \pi(1) = 3c, \quad \pi(2) = 3c, \quad \pi(3) = c$$

SINCE $\pi(0)+\pi(1)+\pi(2)+\pi(3) = 1$, WE HAVE $c = \frac{1}{8}$

THUS

$$\vec{\pi} = \left(\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}\right).$$

THIS SUGGESTS THAT, FOR GENERAL n, WE HAVE

$$\pi(i) = 2^{-n}\binom{n}{i}.$$

TO CHECK THAT THIS GUESS SATISFIES THE DETAILED BALANCE CONDITION

$$\pi(x)p(x,x+1) = 2^{-n}\frac{n!}{x!(n-x)!}\,\frac{n-x}{n}$$

$$= 2^{-n}\frac{n!}{(x+1)!(n-x-1)!}\left(\frac{x+1}{n}\right) = \pi(x+1)p(x+1,x).$$

FOR GENERAL BIRTH AND DEATH CHAINS:

SUPPOSE
$$p(x, x+1) = p_x \qquad x < r$$
$$p(x, x-1) = q_x \qquad x > l$$
$$p(x, x) = r_x \qquad l \leq x \leq r$$

THE DETAILED BALANCE CONDITION

$$\pi(x)\, p_x = \pi(x+1)\, q_{x+1}, \quad \text{THAT IS}$$
$$\pi(x+1) = \frac{p_x}{q_{x+1}}\, \pi(x)$$

IT FOLLOWS THAT

$$\pi(x+2) = \frac{p_{x+1}}{q_{x+2}}\, \pi(x+1) = \frac{p_{x+1}\, p_x}{q_{x+2}\, q_{x+1}}\, \pi(x)$$

IN PARTICULAR WE CAN EXPRESS EACH ENTRY OF $\vec{\pi}$ IN TERMS OF $\pi(l)$:

$$\pi(l+i) = \pi(l) \cdot \frac{p_{l+i-1} \cdot p_{l+i-2} \cdot \ldots \cdot p_l}{q_{l+i} \cdot q_{l+i-1} \cdot \ldots \cdot q_{l+1}}$$

ONE CAN THEN SOLVE FOR $\pi(l)$, SINCE

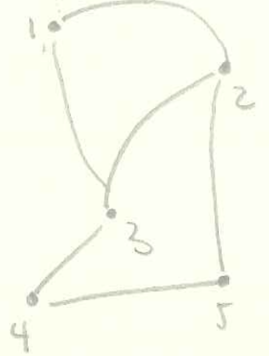
$$\sum_{l \leq x \leq r} \pi(x) = 1.$$

RANDOM WALKS ON GRAPHS.

A <u>GRAPH</u> CONSISTS OF A SET OF VERTICES V, AND AN ADJACENCY MATRIX

$$A = [A(u,v)]$$

WHERE
$$A(u,v) = \begin{cases} 1 & \text{IF THERE IS AN EDGE BETWEEN } u \text{ AND } v. \\ 0 & \text{———.} \end{cases}$$



$V = \{1, 2, 3, 4, 5\}$

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} = A$$

WE DEFINE THE DEGREE OF THE VERTEX $u \in V$ BY THE NUMBER OF EDGES CONNECTED TO u. THAT IS,

$$d(u) = \sum_v A(u,v).$$

IF WE IMAGINE A RANDOM WALKER ON THE GRAPH THAT TAKES A STEP ALONG EACH AVAILABLE EDGE WITH EQUAL PROBABILITY, THEN THE TRANSITION PROBABILITY IS

$$p(u,v) = \frac{A(u,v)}{d(u)}$$

IN OUR EXAMPLE,

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix} = P$$

NOW, SUPPOSE $\pi(u) = c\, d(u)$, THEN

$$\vec{\pi} P(u) = \sum_{v \in S} \pi(v) P(v,u)$$

$$= \sum_{v \in S} c\, d(v) \frac{A(v,u)}{d(v)}$$

$$= c \sum_{v \in S} A(v,u) = c \sum_{v \in S} A(u,v)$$

$$= c\, d(u)$$

THUS $\pi(u) = \dfrac{d(u)}{\sum_v d(v)}$ IS THE STATIONARY DISTRIBUTION.

HW: 1.9.26, 1.9.32, 1.9.48

§1.6 ONE-STEP CALCULATIONS

GAMBLER'S RUIN WITH STATE SPACE $\{0, 1, \ldots, N\}$

LET $V_y = \min\{n \geq 0 : X_n = y\}$

NOTE THIS DIFFERS FROM OUR USUAL TIME OF FIRST RETURN, SINCE WE ALLOW $n = 0$.

NOW LET

$$h(x) = P_x\{V_N < V_0\}$$

THAT IS, $h(x)$ IS THE PROBABILITY THAT, STARTING AT x, WE WIN BEFORE WE GO BUST.

OF COURSE $h(0) = 0$ AND $h(1) = 1$.

TO CALCULATE $h(i)$ FOR $0 < i < N$ WE NOTE THAT

$$h(i) = p\, h(i+1) + (1-p)\, h(i-1)$$

REARRANGE THIS SO THAT

$$h(i) = p\, h(i+1) - p\, h(i) + p\, h(i) + (1-p)\, h(i-1)$$

$$(1-p)\big[h(i) - h(i-1)\big] = p\big[h(i+1) - h(i)\big]$$

THAT IS,

$$h(i+1) - h(i) = \frac{1-p}{p}\big[h(i) - h(i-1)\big]$$

IN THE SPECIAL CASE WHERE $p = \frac{1}{2}$, WE HAVE

$$h(i+1) - h(i) = h(i) - h(i-1)$$

WHICH IMPLIES THAT h HAS CONSTANT SLOPE, AND SINCE $h(0) = 0$, $h(N) = 1$ WE HAVE

$h(x) = \dfrac{x}{N}$. THAT IS

$$P_x\{V_N < V_0\} = \frac{x}{N}$$

MATCHING PENNIES

BOB STARTS WITH 15 PENNIES
CHARLIE STARTS WITH 10 PENNIES

THEY BOTH FLIP A COIN, AND IF THE COINS MATCH BOB GETS A PENNY FROM CHARLIE AND IF THEY DON'T MATCH, CHARLIE GETS A PENNY FROM BOB.

THEY QUIT WHEN ONE OF THEM HAS ALL THE PENNIES.

$15/25$.

WHEN $p \neq 1/2$ THE COMPUTATION TO DETERMINE $h(x)$ IS MORE COMPLICATED.

THE EQUATION

$$h(i+1) - h(i) = \frac{1-p}{p}\left[h(i) - h(i-1)\right]$$

$$\Rightarrow h(i) - h(i-1) = \left(\frac{1-p}{p}\right)^{i-1}\left[h(1) - h(0)\right]$$

LET US SUM THIS EQUATION FROM $i=1$ TO N

$$1 = h(N) - h(0) = \sum_{i=1}^{N} h(i) - h(i-1)$$

$$= \left[h(1) - h(0)\right] \sum_{i=1}^{N} \left(\frac{1-p}{p}\right)^{i-1}$$

FOR CONVENIENCE, SET $\theta = \frac{1-p}{p}$, SO THAT

$$1 = \left[h(1) - h(0)\right] \sum_{i=1}^{N} \theta^{i-1}$$

$$= \left[h(1) - h(0)\right] \frac{1-\theta^N}{1-\theta}$$ IT FOLLOWS THAT

$$h(1) = h(1) - h(0) = \frac{1-\theta}{1-\theta^N}$$

THUS,

$$h(x) = h(x) - h(0) = \sum_{i=1}^{x} h(i) - h(i-1)$$

$$= \frac{1-\theta}{1-\theta^N} \sum_{i=1}^{x} \theta^{i-1} = \frac{1-\theta^x}{1-\theta^N}$$

AND THUS

$$P_x(V_N < V_0) = \frac{\theta^x - 1}{\theta^N - 1} \quad \text{WHERE} \quad \theta = \frac{1-p}{p}.$$

(WHY DOESN'T THIS EQUATION AGREE WITH OUR ANALYSIS WHEN $p = 1/2$?)

LEMMA: SUPPOSE (X_n) IS A MARKOV-CHAIN WITH FINITE STATE SPACE S. LET $a, b \in S$, $C = S \setminus \{a, b\}$ AND SUPPOSE

$$h(a) = 1, \quad h(b) = 0$$

AND

$$h(x) = \sum_y p(x,y)\, h(y).$$

IF $P_x\{\max\{V_a, V_b\} < \infty\} > 0 \quad \forall x \in C$, THEN

$$h(x) = P_x(V_a < V_b).$$

BEFORE PROVING THIS, CONSIDER THE FOLLOWING EXAMPLE

WRIGHT-FISCHER MODEL WITHOUT MUTATION

$$S = \{0, 1, \ldots, N\}$$

$$p(x,y) = \binom{N}{y} \left(\frac{x}{N}\right)^y \left(\frac{N-x}{N}\right)^{N-y}$$

TAKING $a = N$, $b = 0$ AND $h(x) = \frac{x}{N}$, THEN

$$\sum_{y=0}^{N} \binom{N}{y} \left(\frac{x}{N}\right)^{y} \left(\frac{N-x}{N}\right)^{N-y} \frac{y}{N}$$

$$= \sum_{y=0}^{N} \frac{(N-1)!}{(N-y)!\,(y-1)!} \left(\frac{x}{N}\right)^{y} \left(\frac{N-x}{N}\right)^{N-y}$$

$$= \sum_{y=1}^{N} \binom{N-1}{y-1} \left(\frac{x}{N}\right)^{y-1} \left(\frac{N-x}{N}\right)^{N-1-(y-1)} \frac{x}{N}$$

$$= \frac{x}{N} \sum_{y=1}^{N} \binom{N-1}{y-1} \left(\frac{x}{N}\right)^{y-1} \left(\frac{N-x}{N}\right)^{N-1-(y-1)}$$

$$= \frac{x}{N} \underbrace{\sum_{j=0}^{N-1} \binom{N-1}{j} p^{j} (1-p)^{N-1-j}}_{1} \qquad j = y-1, \; p = \frac{x}{N}$$

$$= \frac{x}{N}.$$

THUS $P_x(V_N < V_0) = \frac{x}{N}$

WHICH TELLS US THE PROBABILITY THAT ONE OF THE GENE'S A OR a WILL BECOME EXTINCT IN OUR POPULATION OF N MEMBERS GIVEN THAT WE START WITH x SUCH GENES.

"Pf OF LEMMA":

LET $(\overline{X}_n)$ BE THE CHAIN MODIFIED SO THAT a AND b ARE ABSORBING STATES. THAT IS WE REPLACE

$$p(a,x) = 1 \quad \text{IF} \quad x = a \quad 0 \quad \text{OTHERWISE},$$

$$p(b,x) = 1 \quad \text{IF} \quad x = b \quad 0 \quad \text{OTHERWISE}$$

AND ALL OTHER TRANSITION PROBABILITIES ARE LEFT UNCHANGED.

IF $\overline{V}_a = \min\{n \geq 0 : \overline{X}_n = a\}$ AND

$\overline{V}_b = \min\{n \geq 0 : \overline{X}_n = b\}$

↑ CLEARLY $\overline{V_a} = V_a$ AND $\overline{V_b} = V_b$, AND

$$P\{V_a < V_b\} = P\{\overline{V_a} < \overline{V_b}\}.$$

NOW, BY ASSUMPTION, WE HAVE

$$h(x) = \sum_y p(x,y)\,h(y) \qquad x \in C$$

$$= E_x[h(\overline{X}_1)]$$

IN FACT, BY THE MARKOV PROPERTY, WE HAVE

$$h(x) = E_x[h(\overline{X}_n)]$$

NOW,

$$P_x\{\max\{V_a, V_b\} < \infty\} > 0 \qquad (*)$$

AND THUS THE PEDESTRIAN LEMMA IMPLIES THAT $\overline{X}_n$ WILL EVENTUALLY GET IN a OR b. (NOTE THAT BY $*$ THERE CAN BE NO OTHER ABSORBING STATES).

NOW

$$E_x h(\overline{X}_n) = \frac{1}{n}\sum_{i=1}^{n} E_x h(\overline{X}_i)$$

$$\longrightarrow \sum_y h(y)\,\pi(y) = \underbrace{h(a)}_{1}\;\underbrace{P\{V_a < V_b\}}_{\pi_a}$$

HOWEVER $\frac{1}{n}\sum_{i=1}^{n} E_x h(\overline{X}_i) = \frac{1}{n}\sum_{i=1}^{n} h(x) = h(x)$

WE HAVE $h(x) = P\{V_a < V_b\}$. ✓

DURATION OF FAIR GAMES

CONSIDER THE GAMBLER'S RUIN PROBLEM WITH

$$p(i,i+1) = p(i,i-1) = \frac{1}{2} \qquad 0 < i < N.$$

LET $\tau = \min\{n : X_n \notin (0,N)\}$. THAT IS, τ IS THE TIME WE ENTER AN ABSORBING STATE, AND HAVE TO STOP PLAYING.

LET $h(i) = E_i[\tau]$ BE THE EXPECTED TIME TO AN ABSORBING STATE, STARTING FROM $X_0 = i$. THAT IS, IF WE HAVE i DOLLARS TO START, THE AMOUNT OF TIME WE CAN EXPECT TO PLAY.

CLEARLY $h(0) = h(N) = 0$.

IF $0 < i < N$, THEN

$$h(i) = 1 + \frac{h(i+1) + h(i-1)}{2}$$

THAT IS,

$$(*) \quad h(i+1) - h(i) = -2 + h(i) - h(i-1)$$

AND SUMMING $i = 1$ TO $N-1$, WE HAVE

$$h(N) - h(1) = -2(N-1) + h(N-1) - h(0)$$

THAT IS, $-h(1) = -2(N-1) + h(N-1)$

BY SYMMETRY $h(1) = h(N-1)$ (WHY?), AND THUS

$$h(1) = h(N-1) = N-1.$$

NOW, SUMMING $*$ FROM $i=1$ TO $j-1$

$$h(j) - h(1) = -2(j-1) + h(j-1) - h(0)$$

THAT IS $h(j) - h(j-1) = N-1-2(j-1)$

AND, SUMMING THIS FROM $j=1$ TO x

$$h(x) - h(0) = x(N-1) - 2\sum_{j=1}^{x}(j-1)$$

$$= x(N-1) - x(x-1) = x(N-x).$$

THAT IS,

$$E_i[\tau] = i(N-i).$$

EX: 1.6.1

<u>THM</u>: IF P IS AN IRREDUCIBLE TRANSITION MATRIX WITH STATIONARY DISTRIBUTION π, THEN

$$\pi(y) = \frac{1}{E_y[T_y]}$$

TIME OF FIRST RETURN TO y

IDEA OF PROOF:

IF $N_n(y)$ IS THE NUMBER OF VISITS TO y BY TIME n, THEN

$$\frac{N_n(y)}{n} \longrightarrow \pi(y)$$

ON THE OTHER HAND, THE TIMES BETWEEN RETURNS TO y ARE INDEPENDENT WITH MEAN $E_y[T_y]$, THUS ON AVERAGE WE MAKE 1 VISIT TO y EVERY $E_y[T_y]$ TURNS

THUS WE EXPECT

$$\frac{N_n(y)}{n} \longrightarrow \frac{1}{E_y[T_y]} = \pi(y).$$

✓

WAITING TIME FOR HT

SUPPOSE WE START FLIPPING A COIN AND STOP WHEN WE FIRST COMPLETE THE PATTERN HT.

HOW LONG SHOULD WE EXPECT THIS TO TAKE?

CONSIDER THE MARKOV CHAIN WITH TRANSITION MATRIX

$$\begin{array}{c|cccc} & HH & HT & TH & TT \\ \hline HH & 1/2 & 1/2 & 0 & 0 \\ HT & 0 & 0 & 1/2 & 1/2 \\ TH & 1/2 & 1/2 & 0 & 0 \\ TT & 0 & 0 & 1/2 & 1/2 \end{array}$$

THIS MATRIX IS DOUBLY STOCHASTIC (IRREDUCIBLE AND APERIODIC) AND THUS THE STATIONARY DISTRIBUTION IS

$$(1/4, 1/4, 1/4, 1/4)$$

IT FOLLOWS THAT

$$E(T_{HT} \mid X_0 = HT) = E_{HT} T_{HT} = \frac{1}{\pi(HT)} = 4$$

NOTICE THAT STARTING WITH NOTHING IS THE SAME AS STARTING AT HT, THAT IS

$$P\{HT \text{ FOR THE FIRST TIME ON ROUND } n \geq 1\}$$

IS THE SAME AS

$$P\{HT \text{ FOR THE FIRST TIME ON ROUND } n \geq 1 \mid X_0 = HT\}$$

AND THUS $E[T_{HT}] = 4.$

HW: 1.9.39, 1.9.41

§ 1.7: INFINITE STATE SPACES

IMAGINE A PARTICLE THAT MOVES ON $\{0, 1, 2, \ldots\}$ WITH

$$p(i, i+1) = p \qquad i \geq 0$$

$$p(i, i-1) = 1-p \qquad i \geq 1$$

$$p(0,0) = 1-p \qquad \text{(CANT JUMP LEFT OF 0.}$$

THIS IS A BIRTH AND DEATH CHAIN, AND IF WE WERE TO USE THE DETAILED BALANCE EQUATION TO DERIVE THE STATIONARY DISTRIBUTION, WE WOULD FIND

$$p\,\pi(i) = (1-p)\,\pi(i+1) \qquad i \geq 0$$

THAT IS,

$$\pi(i+1) = \pi(i)\left(\frac{p}{1-p}\right),$$

AND SETTING $\pi(0) = c$

$$\pi(i) = c\left(\frac{p}{1-p}\right)^i$$

WE HAVE THREE POSSIBILITIES:

i) $p > \frac{1}{2}$: $\frac{p}{1-p} > 1$ AND $\pi(i)$ GROWS EXPONENTIALLY.

THIS CLEARLY CAN'T HAPPEN SINCE WE NEED $\sum_{i=0}^{\infty} \pi(i) = 1$, THUS NO π EXISTS

ii) $p = \frac{1}{2}$: $\frac{p}{1-p} = 1$ AND $\pi(i) = c \quad \forall i$

THIS CANNOT HAPPEN, FOR THE SAME REASON, AND THUS NO STATIONARY DISTRIBUTION EXISTS.

iii) $p < \frac{1}{2}$: SET $\theta = \frac{p}{1-p} < 1$, AND

$$1 = c\sum_{i=0}^{\infty} \theta^i = \frac{c}{1-\theta} \implies c = 1-\theta$$

THUS $\pi(i) = (1-\theta)\,\theta^i$

THIS IS THE UNIQUE STATIONARY DISTRIBUTION, SINCE THE MARKOV CHAIN IS IRREDUCIBLE.

CLASSIFYING STATES:

I. WHEN $p > \frac{1}{2}$ ALL STATES ARE TRANSIENT

II. WHEN $p = \frac{1}{2}$:

LET $V_y = \min\{n \geq 0 : X_n = y\}$.

FOR ANY $N > 0$ WE HAVE

$$P_x\{V_N < V_0\} = \frac{x}{N}$$

FIXING x AND LETTING $N \to \infty$, WE HAVE

$$P_x\{V_N < V_0\} \longrightarrow 0$$

AND THUS

$$P_x\{V_0 < \infty\} = 1$$

NOW, LET $\tau_N = \min\{n : X_n \notin \{0, N\}\}$

CLEARLY $\tau_N \leq V_0$, AND

$$E_1 \tau_N = N - 1$$

THUS $E_1[V_0] \geq N-1 \longrightarrow \infty$ AS $N \to \infty$.

THAT THE EXPECTED TIME OF RETURN TO 0, GIVEN WE START AT 1 IS INFINITE, EVEN THOUGH WE WILL RETURN TO 0 WITH PROBABILITY 1. (!)

THAT IS: $P_0\{T_0 < \infty\} = 1$ BUT $E_0 T_0 = \infty$.

<u>DEF</u>: IF A STATE x IS SUCH THAT $P_x\{T_x < \infty\} = 1$ AND $E_x[T_x] < \infty$ THEN WE SAY x IS <u>POSITIVE RECURRENT</u>. IF $P_x\{T_x < \infty\} = 1$ BUT $E_x[T_x] = \infty$ THEN WE SAY x IS <u>NULL RECURRENT</u>.

THM: FOR AN IRREDUCIBLE CHAIN THE FOLLOWING ARE EQUIVALENT.

i) SOME STATE IS POSITIVE RECURRENT

ii) ALL STATES ARE POSITIVE RECURRENT

iii) THERE IS A STATIONARY DISTRIBUTION π.

THE CONVERGENCE THEOREM REVISITED:

THM. SUPPOSE p IS IRREDUCIBLE, APERIODIC AND ALL STATES ARE POSITIVE RECURRENT. THEN

$$\lim_{n \to \infty} p^n(x, y) = \pi(y)$$

WHERE π IS THE UNIQUE NON-NEGATIVE SOLUTION TO

$$\vec{\pi} P = \vec{\pi} \quad \text{WITH} \quad \sum_i \pi(i) = 1.$$

BRANCHING PROCESSES:

CONSIDER A POPULATION IN WHICH EACH INDIVIDUAL IN THE n^{TH} GENERATION GIVES BIRTH TO AN INDEPENDENT AND IDENTICALLY DISTRIBUTED NUMBER OF CHILDREN.

WE SET X_n TO BE THE NUMBER OF INDIVIDUALS AT TIME n.

IF THE # OF CHILDREN OF EACH PARENT IS A RANDOM VARIABLE IN $Y_1, Y_2, \ldots$ WITH

$$P\{Y_m = k\} = p_k \quad \leftarrow \text{THE PROBABILITY A PARENT HAS } k \text{ CHILDREN}$$

THEN $p(i,j) = P(Y_1 + \ldots + Y_i) = j$ FOR $i > 0$ $j \geq 0$

CLEARLY $p(0,0) = 1$ IS AN ABSORBING STATE.

Q: WHAT IS THE PROBABILITY THE SPECIES AVOIDS EXTINCTION?

THE MEAN OF EACH Y_m IS GIVEN BY

$$\mu = \sum_{k=0}^{\infty} k p_k$$

AND

$$E[X_n \mid X_{n-1}] = \mu X_{n-1}$$

NOW, TAKING EXPECTATIONS OF BOTH SIDES,

$$E[\, E[X_n \mid X_{n-1} = y]]$$

$$= E\left[\sum_{x \in S} x \, \frac{P\{X_n = x, X_{n-1} = y\}}{P\{X_{n-1} = y\}} \right]$$

$$= E\left[\sum_{x \in S} \sum_{y \in S} x \, \frac{P\{X_n = x, X_{n-1} = y\}}{P\{X_{n-1} = y\}} P\{X_{n-1} = y\} \right]$$

$$= E\left[\sum_{x \in S} x \sum_{y \in S} P\{X_n = x, X_{n-1} = y\} \right]$$

$$= E[X_n]$$

THE RIGHT HAND SIDE IS $\mu E[X_{n-1}]$.
THUS

$$E[X_n] = \mu E[X_{n-1}] = \mu^n E[X_0]$$

IF $\mu < 1$, THEN $E[X_n] \to 0$ EXPONENTIALLY FAST.
AND, SINCE

$$E[X_n] \geq P\{X_n \geq 1\}$$

WE HAVE $P\{X_n \geq 1\} \to 0$, AND THUS

I) IF $\mu < 1$, EXTINCTION OCCURS WITH PROBABILITY 1.

EX: 1.7.2

II) IF $\mu \geq 1$, LET'S DO A ONE-STEP CALCULATION:

LET ρ BE THE PROBABILITY THAT THIS PROCESS DIES OUT STARTING FROM $X_0 = 1$.

IF THERE ARE k CHILDREN IN THE FIRST GENERATION THEN IN ORDER FOR EXTINCTION TO OCCUR, THE FAMILY LINE OF EACH CHILD MUST DIE OUT, WHICH OCCURS WITH PROBABILITY ρ^k.

THUS, $\rho = \sum_{k=0}^{\infty} p_k \rho^k$ (*)

NOTICE THAT THE RIGHT HAND SIDE IS THE PROBABILITY GENERATING FUNCTION OF ONE OF THE Y_m (!)

(SEE SECTION 0.3 FOR MORE ABOUT PGF).

LET $\varphi(\rho) = \sum_{k=0}^{\infty} p_k \rho^k$, SO THAT (*) CAN BE WRITTEN AS:

$$\rho = \varphi(\rho).$$

NOTICE THAT THIS EQUATION IS TRIVIALLY SATISFIED WHEN $\rho = 1$, SINCE

$$\sum_{k=0}^{\infty} p_k = 1.$$

LEMMA: THE EXTINCTION PROBABILITY ρ IS THE SMALLEST SOLUTION OF THE EQUATION $\varphi(x) = x$ WITH $0 \le x \le 1$.

Pf: IN ORDER FOR THE PROCESS TO HIT 0 BY TIME n, ALL OF THE PROCESSES STARTED BY FIRST GENERATION INDIVIDUALS MUST HIT 0 BY TIME $n-1$. THUS

$$P\{X_n = 0\} = \sum_{k=0}^{\infty} p_k \, P\{X_{n-1} = 0\}^k$$

THUS, IF WE DENOTE $\rho_n = P\{X_n = 0\}$ WE HAVE

$$\rho_n = \varphi(\rho_{n-1}); \quad n \ge 1. \qquad (**)$$

SINCE 0 IS AN ABSORBING STATE

$$\rho_0 \le \rho_1 \le \rho_2 \le \dots$$

AND THE SEQUENCE CONVERGES TO A LIMIT ρ_∞ (WHY?)

TAKING LIMITS OF BOTH SIDES (AND ASSUMING THAT φ IS CONTINUOUS), WE FIND

$$\rho_\infty = \varphi(\rho_\infty)$$

NOW, SUPPOSE ρ IS THE SMALLEST SOLUTION TO $x = \varphi(x)$. CLEARLY

$$\rho_0 = 0 \le \rho$$

USING THE FACT THAT φ IS INCREASING (WHY?)

THEN

$$\rho_1 = \varphi(\rho_0) \le \varphi(\rho) = \rho$$

REPEATING THIS, WE MUST HAVE

$$\rho_n \le \rho \quad \text{FOR ALL } n$$

TAKING LIMITS WE MUST HAVE $\rho_\infty \le \rho$,

BUT SINCE ρ_∞ IS A SOLUTION OF $x = \varphi(x)$ AND ρ IS THE SMALLEST SOLUTION, IN FACT WE MUST HAVE

$$\rho_\infty = \rho. \checkmark$$

THUS WE HAVE A METHOD FOR FINDING THE PROBABILITY OF EXTINCTION FROM THE PGF OF THE Y's.

BINARY BRANCHING

SUPPOSE $p_0 = 1-a$, $p_2 = a$

IN THIS CASE WE HAVE

$$\varphi(x) = ax^2 + 1 - a$$

SOLVING $\varphi(x) = x$ WE FIND

$$ax^2 - x + 1 - a = (x-1)(ax - (1-a))$$

THUS THE SOLUTIONS ARE

$$x = 1 \quad \text{AND} \quad x = \frac{1-a}{a}$$

IF $a \le \frac{1}{2}$ THE SMALLEST SOLUTION IS 1, WHILE IF

$a > \frac{1}{2}$ THE SMALLEST SOLUTION IS $\frac{1-a}{a}$

AND THUS IN THIS CASE THERE IS A POSITIVE PROBABILITY OF AVOIDING EXTINCTION.

THM: IF (X_n) IS A BRANCHING PROCESS WITH

$$p(i,j) = P\{Y_1 + \dots + Y_i = j\}$$

WHERE THE Y_m ARE i.i.d RANDOM VARIABLES WITH MEAN μ, THEN

i) IF $\mu < 1$ EXTINCTION OCCURS WITH PROBABILITY 1.

ii) IF $\mu > 1$ THERE IS A POSITIVE PROBABILITY OF AVOIDING EXTINCTION.

Pf: WE HAVE ALREADY PROVEN i).

TO PROVE ii) IT SUFFICES TO SHOW THERE IS A SOLUTION TO

$$\varphi(x) = x \quad \text{LESS THAN } 1.$$

RECALL

$$\varphi(x) = \sum_{k=0}^{\infty} p_k x^k$$

AND THUS, IF $p_0 = 0$ THE SMALLEST SOLUTION IS $x = 0$ AND THERE IS NO PROBABILITY OF EXTINCTION.

IF $p_0 > 0$ THEN $\varphi(0) = p_0 > 0$

DIFFERENTIATING φ WE HAVE

$$\varphi'(x) = \sum_{k=1}^{\infty} p_k \, k x^{k-1}$$

$$\varphi'(1) = \sum_{k=1}^{\infty} k p_k = \mu$$

THUS, IF WE DEFINE $\psi(x) = \varphi(x) - x$

WE HAVE

$$\psi(1) = 0 \quad \text{AND} \quad \psi'(1) = \mu - 1 > 0$$

THIS MEANS $\exists \ \epsilon > 0$ SUCH THAT $\psi(1-\epsilon) < 0$

AND, SINCE $\psi(0) = p_0 > 0$ BY THE INTERMEDIATE VALUE THEOREM WE HAVE

$$\exists \ x \in (0, 1-\epsilon) \text{ SO THAT } \psi(x) = 0. \ \checkmark$$

THM (CONTINUED)

iii) IF $\mu = 1$ AND WE EXCLUDE THE TRIVIAL CASE $p_1 = 1$, THEN EXTINCTION OCCURS WITH PROBABILITY 1.

Pf: WE NEED ONLY SHOW THAT THERE IS NO ROOT OF $\varphi(x)$ BETWEEN 0 AND 1.

IF $p_1 < 1$, THEN IF $y \in (0,1)$

$$\varphi'(y) = \sum_{k=1}^{\infty} p_k \cdot k x^{k-1} < \sum_{k=1}^{\infty} p_k k = 1$$

THUS,

$$\varphi(x) = \varphi(1) - \int_x^1 \varphi'(y)\,dy > 1 - (1-x) = x.$$

THAT IS $\varphi(x) > x$ FOR ALL $x < 1$. ✓

HW: 1.9.52, 1.9.53

CHAPTER 2: MARTINGALES

DEF: SUPPOSE A IS AN EVENT IN SOME SAMPLE SPACE. WE DEFINE THE INDICATOR FUNCTION OF A BY

$$1_A(x) = \begin{cases} 1 & \text{IF } x \in A \\ 0 & \text{IF } x \in A^c \end{cases}$$

AND, GIVEN A RANDOM VARIABLE WE DEFINE THE INTEGRAL OF Y OVER A BY

$$E[Y;A] = E[Y 1_A].$$

NOTICE THAT, IF f_Y IS THE DENSITY OF Y (AND Y IS A CONTINUOUS RANDOM VARIABLE,

$$E[Y;A] = E[Y 1_A] = \int_{\mathbb{R}} 1_A(y)\, y\, f_Y(y)\,dy$$

$$= \int_A y\, f_Y(y)\,dy,$$ WHICH EXPLAINS THE TERMINOLOGY.

DEF: WE DEFINE THE CONDITIONAL EXPECTATION OF Y GIVEN A TO BE

$$E[Y|A] = E[Y;A]/P(A).$$

THIS IS THE EXPECTED VALUE OF $P\{\cdot|A\}$, THAT IS THE EXPECTED VALUE OF RESTRICTING Y TO A (AND NORMALIZING TO MAKE THIS A RANDOM VARIABLE WITH STATE SPACE A).

CLEARLY, IF Y AND Z ARE RANDOM VARIABLES

$$E[Y+Z;A] = E[Y;A] + E[Z;A]$$

AND

$$E[Y+Z|A] = E[Y|A] + E[Z|A].$$

LEMMA: $E[cY|A] = cE[Y|A]$ FOR ANY CONSTANT c.

LEMMA: IF B IS THE DISJOINT UNION OF $A_1, \ldots, A_k$, THEN

$$E[Y;B] = \sum_{j=1}^{k} E[Y;A_j].$$

THIS FOLLOWS SINCE, IN THIS CASE,

$$1_B(x) = \sum_{j=1}^{k} 1_{A_j}(x).$$

LEMMA: IF B IS THE DISJOINT UNION OF $A_1, \ldots, A_k$, THEN

$$E[Y|B] = \sum_{j=1}^{k} E[Y|A_j] \cdot \frac{P(A_j)}{P(B)}$$

PF:

$$E[Y|B] = \frac{E[Y;B]}{P(B)} = \sum_{j=1}^{k} \frac{E[Y;A_j]}{P(B)}$$

$$= \sum_{j=1}^{k} \frac{E[Y;A_j]}{P(A_j)} \frac{P(A_j)}{P(B)} = \sum_{j=1}^{k} E[Y|A_j] \frac{P(A_j)}{P(B)} \checkmark$$

§ 2.2 : EXAMPLES OF MARTINGALES

DEF : A SEQUENCE OF RANDOM VARIABLES (M_n) IS CALLED A MARTINGALE IF

$$\text{i) } E[|M_n|] < \infty,$$

AND FOR ANY POSSIBLE VALUES $m_n, \ldots, m_0$

$$\text{ii) } E[M_{n+1} \mid M_n = m_n, M_{n-1} = m_{n-1}, \ldots, M_0 = m_0] = m_n.$$

THINK OF A GAMBLER WHOSE WINNINGS AT TIME n ARE M_n, THE SECOND CONDITION SAYS THAT THE AVERAGE AMOUNT OF MONEY THE GAMBLER HAS AT TIME $n+1$ IS WHAT HE HAS AT TIME n.

TO RELATE THIS TO FAIR GAMES, WE CAN REWRITE ii) AS

$$E[M_{n+1} - M_n \mid M_n = m_n, M_{n-1} = m_{n-1}, \ldots, M_0 = m_0] = 0$$

WHICH IS SENSIBLE.

THM: LET $\xi_1, \xi_2, \ldots$ BE INDEPENDENT RANDOM VARIABLES WITH

$$E[\xi_i] = 0$$

THEN $M_n = M_0 + \xi_1 + \ldots + \xi_n$ DEFINES A MARTINGALE.

Pf : NOTICE THAT

$$M_{n+1} - M_n = \xi_{n+1}$$

WHICH IS INDEPENDENT OF $M_0, \ldots, M_n$. THUS

$$E[M_{n+1} - M_n \mid M_n = m_n, \ldots, M_0 = m_0] = E[\xi_{n+1}] = 0. \checkmark$$

MOST CASINO GAMES ARE NOT FAIR, WHICH MOTIVATE THE FOLLOWING DEFINITIONS

DEF: WE SAY (Y_n) IS A SUPERMARTINGALE IF

$$E[Y_{n+1} - Y_n \mid Y_n = y_n, \dots, Y_0 = y_0] \leq 0$$

(THAT IS IF THE GAME FAVOURS THE CASINO), AND A SUBMARTINGALE IF

$$E[Y_{n+1} - Y_n \mid Y_n = y_n, \dots, Y_0 = y_0] \geq 0.$$

BRANCHING PROCESSES:

LET Z_n BE THE # OF CHILDREN IN GENERATION n WITH THE PROBABILITY OF A SINGLE INDIVIDUAL HAVING k CHILDREN BEING p_k. LET

$$\mu = \sum_{k=0}^{\infty} k p_k,$$

THEN IT IS CLEAR

$$E[Z_{n+1} \mid Z_n = z_n, \dots, Z_0 = z_0] = \mu z_n$$

SO (Z_n) IS A SUPER MARTINGALE IF $\mu \leq 1$

A MARTINGALE IF $\mu = 1$

A SUBMARTINGALE IF $\mu \geq 1$.

NOTICE THAT (M_n) WITH $M_n = \dfrac{Z_n}{\mu}$ IS A MARTINGALE.

STOCK PRICES

LET $\xi_1, \xi_2, \dots$ BE INDEPENDENT POSITIVE RANDOM VARIABLES WITH

$$E[\xi_i] < \infty, \text{ AND LET } X_n = X_0 \xi_1 \cdots \xi_n$$

HERE WE THINK OF $\xi_n - 1$ AS THE CHANGE IN VALUE OF A STOCK PRICE OVER A FIXED TIME INTERVAL AS A FRACTION OF ITS CURRENT VALUE.

THE MULTIPLICATIVE FORMULATION IS USEFUL SINCE IT GUARENTEES THAT X_n IS POSITIVE, AND EMPIRICAL EVIDENCE SUGGESTS THAT THE FLUCTUATIONS IN A STOCK PRICE IS PROPORTIONAL TO ITS PRICE

CONCRETE EXAMPLE

DISCRETE BLACK-SCHOLES MODEL:

$$\xi_i = e^{\eta_i} \quad \text{WHERE} \quad \eta_i = N(\mu, \sigma^2)$$

WRITING $X_{n+1} = X_n \xi_{n+1}$ THEN

$$\begin{aligned} E[X_{n+1} \mid X_n = x_n, \ldots, X_0 = x_0] &= E[x_n \xi_{n+1} \mid X_n = x_n, \ldots, X_0 = x_0] \\ &= x_n E[\xi_{n+1} \mid X_n = x_n, \ldots, X_0 = x_0] \\ &= x_n E[\xi_{n+1}] \quad (\text{WHY?}) \ldots \end{aligned}$$

THUS (X_n) IS A SUBMARTINGALE IF $E[\xi_i] > 1$ A MARTINGALE IF $E[\xi_i] = 1$ AND A SUPERMARTINGALE IF $E[\xi_i] < 1$.

THM: IF (Y_m) IS A SUPERMARTINGALE THEN $E[Y_m] \geq E[Y_n]$ IF $m \leq n$.

Pf: IT SUFFICES TO SHOW $E[Y_k] \geq E[Y_{k+1}]$.

TO DO THIS WRITE

$$\vec{i} = (i_n, i_{n-1}, \ldots, i_0)$$

AND LET

$$A_{\vec{i}} = \{Y_n = i_n, Y_{n-1} = i_{n-1}, \ldots, Y_0 = i_0\}$$

NOW,

$$\begin{aligned} E[Y_{k+1} - Y_k] &= \sum_{\vec{i}} E[Y_{k+1} - Y_k ; A_i] \\ &= \sum_{i} P(A_i) \underbrace{E[Y_{k+1} - Y_k \mid A_i]}_{\leq 0 \text{ BY ASSUMPTION.}} \leq 0 \end{aligned}$$

✓

THE SAME PROOF WORKS TO SHOW

THM: IF (Y_m) IS A SUBMARTINGALE AND $0 \le m < n$, THEN

$$E[Y_m] \le E[Y_n]$$

COR: IF (M_m) IS A MARTINGALE THEN $E[M_m] = E[M_n]$ FOR ALL m, n.

WRIGHT-FISHER MODEL WITH NO MUTATION:

$$p(x,y) = \binom{N}{y} \left(\frac{x}{N}\right)^y \left(\frac{N-x}{N}\right)^{N-y}$$

NOW

$$E[X_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0]$$

$$= \sum_{k=0}^{N} k \, P\{X_{n+1} = k \mid X_n = x_n, \ldots, X_0 = x_0\}$$

$$= \sum_{k=0}^{N} k \, P\{X_{n+1} = k \mid X_n = x_n\}$$

$$= \sum_{k=0}^{N} k \binom{N}{k} \left(\frac{x_n}{N}\right)^k \left(\frac{N-x_n}{N}\right)^{N-k}$$

WHICH IS THE EXPECTATION OF THE BINOMIAL RANDOM VARIABLE WITH N TRIALS AND PROBABILITY $\frac{x_n}{N}$,

THAT IS,

$$E[X_{n+1} \mid X_n = x_n, \ldots, X_0 = x_0] = N \cdot \frac{x_n}{N} = x_n.$$

THUS (X_n) IS A MARTINGALE.

THM: LET (X_n) BE A MARKOV CHAIN WITH TRANSITION PROBABILITY p, AND LET $h(x,n)$ BE A FUNCTION SO THAT

$$h(x,n) = \sum_y p(x,y)\, h(y, n+1),$$

THEN $(M_n) = (h(X_n, n))$ IS A MARTINGALE.

Pf: BY THE MARKOV PROPERTY

$$E[h(X_{n+1}, n+1) \mid X_n = x_n, \ldots, X_0 = x_0]$$
$$= E[h(X_{n+1}, n+1) \mid X_n = x_n]$$
$$= \sum_{y \in S} h(y, n+1)\, p(x_n, y) = h(x_n, n)$$

THIS IS NOT QUITE WHAT WE WANT, SINCE WE WANT

$$E[M_{n+1} \mid M_n = m_n, \ldots, M_0 = m_0] = m_n$$

FIX, $m_n, \ldots, m_0$ AND LET B BE THE EVENT

$$B = \{M_n = m_n, \ldots, M_0 = m_0\}$$

SINCE GIVEN $\vec{i} = (i_n, \ldots, i_0)$ LET $A_{\vec{i}}$ BE THE EVENT

$$A_i = \{X_n = i_n, \ldots, X_0 = i_0\}$$

AND LET I BE THE SET OF $\vec{i}$ SUCH THAT

$$f(i_n, n) = m_n, \ldots, f(i_0, 0) = m_0$$

THAT IS

$$B = \bigcup_{\vec{i} \in I} A_{\vec{i}}.$$ THIS IS A DISJOINT UNION.

NOW

$$E[M_{n+1} \mid B] = \sum_{\vec{i} \in I} E[M \mid A_i] \frac{P\{A_i\}}{P\{B\}}$$

$$= m_n \sum_{\vec{i} \in I} \frac{P\{A_i\}}{P\{B\}} = m_n$$

WHICH IS WHAT WE WANTED TO SHOW. ✓

THIS MOTIVATES THE FOLLOWING DEFINITION

DEF: WE SAY (M_n) IS A MARTINGALE WITH RESPECT TO (X_n) IF

$$E[|M_n|] < \infty$$

AND FOR ANY POSSIBLE VALUES $x_n, \dots, x_0$

$$E[M_{n+1} - M_n \mid X_n = x_n, \dots, X_0 = x_0] = 0.$$

FOR ALL OF OUR EXAMPLES WE WILL HAVE

$$M_n = h(X_n, n)$$ AS BEFORE WHERE (X_n) IS A MARKOV CHAIN.

VARIANCE MARTINGALE

LET $\xi_1, \xi_2, \dots$ BE INDEPENDENT WITH $E[\xi_i] = 0$ AND

$$\text{VAR}(\xi_i) = E[\xi_i^2] = \sigma_i^2$$

LET $S_n = S_0 + \xi_1 + \dots + \xi_n$ WHERE S_0 IS A CONSTANT. AND LET

$$v_n = \sum_{i=1}^{n} \sigma_i^2$$ BE THE VARIANCE OF S_n.

THEN $M_n = S_n^2 - v_n$ IS A MARTINGALE WITH RESPECT TO S_n.

Pf: $v_{n+1} - v_n = \sigma_{n+1}^2$, SO

$$M_{n+1} - M_n = (S_n + \xi_{n+1})^2 - S_n^2 - \sigma_{n+1}^2$$
$$= 2\xi_{n+1} S_n + \xi_{n+1}^2 - \sigma_{n+1}^2$$

LETTING $A = \{S_n = s_n, \dots, S_0 = s_0\}$, AND SINCE $S_n = s_n$ ON A,

$$E[2\xi_{n+1} S_n \mid A] = 2 s_n E[\xi_{n+1} \mid A] = 0$$

SINCE ξ_{n+1} IS INDEPENDENT OF $S_0, \dots, S_n$, AND HENCE OF A.

IT FOLLOWS THAT

$$E[M_{n+1} - M_n] = E[\xi_{n+1}^2 - \sigma_{n+1}^2] = 0,$$

AND, SINCE

ξ_{n+1} IS INDEPENDENT OF A, WE HAVE

$$E[M_{n+1} - M_n \mid A] = E[\xi_{n+1}^2 - \sigma_{n+1}^2 \mid A] = 0.$$ ✓

GAMBLER'S RUIN

SUPPOSE $\xi_1, \xi_2, \ldots$ ARE INDEPENDENT WITH

$$P\{\xi_i = 1\} = p \text{ AND } P\{\xi_i = -1\} = q = 1-p,$$

AND LET $S_n = S_0 + \xi_1 + \ldots + \xi_n$ AND

$$g(x) = \left(\frac{1-p}{p}\right)^x$$

THEN $M_n = g(S_n)$ IS A MARTINGALE WITH RESPECT TO S_n.

__Pf__: SINCE (S_n) IS A MARKOV CHAIN (WHY?) IT SUFFICES TO CHECK

$$g(x) = \sum_y p(x,y)\, g(y)$$

THAT IS,

$$\sum_y p(x,y) g(y) = p\left(\frac{1-p}{p}\right)^{x+1} + (1-p)\left(\frac{1-p}{p}\right)^{x-1}$$

$$= (1-p)\left(\frac{1-p}{p}\right)^x + p\left(\frac{1-p}{p}\right)^x = \left(\frac{1-p}{p}\right)^x$$

AS DESIRED. ✓

LIKELIHOOD RATIOS

SUPPOSE WE ARE TESTING THE HYPOTHESIS THAT OBSERVATIONS $\xi_1, \xi_2, \ldots$ ARE INDEPENDENT AND HAVE DENSITY FUNCTION f, BUT IN FACT THEY ARE IN FACT INDEPENDENT AND HAVE DENSITY FUNCTION g WHERE

$$\{x : f(x) > 0\} = \{x : g(x) > 0\}$$

LET

$$h(x) = \begin{cases} f(x)/g(x) & \text{WHEN } g(x) > 0 \\ 0 & \text{IF } g(x) = 0 \end{cases}$$

THEN $M_n = h(\xi_1)\,h(\xi_2)\cdots h(\xi_n)$ IS A MARTINGALE.

POLYA'S URN SCHEME

SUPPOSE AN URN CONTAINS RED AND GREEN BALLS. AT TIME 0 THERE IS ONE BALL OF EACH COLOUR. AT TIME n WE DRAW OUT A BALL CHOSEN AT RANDOM. WE RETURN IT TO THE URN AND ADD ONE MORE BALL OF THE COLOUR CHOSEN. LET X_n BE THE FRACTION OF RED BALLS AT TIME n.

NOTICE AT TIME n THERE ARE $n+2$ BALLS AND, IF $X_n = x_n$ THERE ARE $(n+2)x_n$ RED BALLS.

THUS,

$$E[X_{n+1} \mid X_n = x_n, \ldots, X_0 = x_0]$$

$$= x_n \left(\frac{(n+2)x_n + 1}{n+3} \right) + (1 - x_n) \frac{(n+2)x_n}{n+3}$$

$$= \frac{x_n}{n+3} + \frac{(n+2)x_n}{n+3} = x_n$$

AND SO X_n IS A MARTINGALE.

HW: 2.5.2, 2.5.6, 2.5.7

§ 2.3: OPTIONAL STOPPING THEOREM

SUPPOSE WE ARE PLAYING A GAME IN WHICH YOU EITHER WIN OR LOSE THE AMOUNT YOU BET.

ONE STRATEGY IS THAT IF YOU WIN YOU BET $1 ON THE NEXT PLAY, AND IF YOU LOSE YOU DOUBLE YOUR PREVIOUS BET.

THUS IF YOU LOSE THREE TIMES IN A ROW AND THEN WIN WE HAVE

	L	L	L	W
BET	1	2	4	8
NET PROFIT	-1	-3	-7	1

THIS SYSTEM SEEMS TO GUARENTEE WE WILL NET MONEY ASSUMING THERE IS SOME POSITIVE PROBABILITY OF WINNING.

WHAT'S THE CATCH?

THM: SUPPOSE WE USE THE DOUBLING STRATEGY ON A SUPERMARTINGALE UP TO A FIXED TIME n AND WE LET W_n BE OUR NET WINNINGS. THEN

$$E[W_n] \leq 0$$

WE NEED TO MAKE THIS PRECISE, AND WE WILL GENERALIZE THE SITUATION.

LET H_n DENOTE THE AMOUNT WE BET ON n^{TH} GAME, AND DENOTE THE OUTCOME OF THE n^{TH} GAME BY X_n. A BETTING STRATEGY IS A WAY OF MAKING H_n A FUNCTION OF $X_0, \ldots, X_{n-1}$

DEF: LET $\mathcal{F}(X_0, \ldots, X_m)$ BE THE SET OF RANDOM VARIABLES WHICH ARE FUNCTIONS OF $X_0, \ldots, X_m$.

OUR BETTING STRATEGY THUS CAN BE WRITTEN

$$H_n \in \mathcal{F}(X_0, \ldots, X_{n-1})$$

TO SEE HOW OUR NET WINNINGS ARE CALCULATED FROM OUR BETTING STRATEGY, SUPPOSE

$$\xi_m = \begin{cases} 1 & \text{IF WE WIN ON THE } m^{TH} \text{ GAME} \\ -1 & \text{IF WE LOSE} \end{cases}$$

THEN $W_n = W_0 + \sum_{m=1}^{n} H_m \xi_m$

IT IS CONVENIENT TO SET

$$Y_n = Y_0 + \xi_1 + \dots + \xi_n$$

BE THE WINNINGS OF A GAMBLER WHO ALWAYS BETS \$1.
THEN

$$\xi_n = Y_n - Y_{n-1}, \text{ AND}$$
$$W_n = W_0 + \sum_{m=1}^{n} H_m (Y_m - Y_{m-1})$$

<u>THM</u>: SUPPOSE (Y_n) IS A SUPERMARTINGALE WITH RESPECT TO (X_n) AND THAT

$$H_n \in \mathcal{F}(X_0, \dots, X_{n-1})$$

SATISFIES $0 \le H_n \le c_n$ WHERE c_n IS A CONSTANT THAT MAY DEPEND ON n.

THEN, $W_n = W_0 + \sum_{m=1}^{n} H_m (Y_m - Y_{m-1})$ IS A SUPERMARTINGALE. W.R.T. (X_n)

<u>Pf</u>: THE GAIN AT TIME $n+1$ IS

$$W_{n+1} - W_n = H_{n+1}(Y_{n+1} - Y_n)$$

LET $A = \{X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0\}$. SINCE H_{n+1} IS A FUNCTION OF $X_0, X_1, \dots, X_n$ IT IS CONSTANT ON THE EVENT A, AND

$$E[H_{n+1}(Y_{n+1} - Y_n) \mid A]$$
$$= H_{n+1} E[Y_{n+1} - Y_n \mid A] \le 0$$

SINCE, $W_0, \ldots, W_n$ ARE FUNCTIONS OF $X_0, \ldots, X_n$ WE CAN USE THE PREVIOUS TRICK TO SHOW THAT IN FACT (W_n) IS A SUPERMARTINGALE. ✓

IT FOLLOWS NOW THAT $E[W_n] \le E[W_0] = 0$

<u>DEF</u>: WE SAY T IS A <u>STOPPING TIME</u> WITH RESPECT TO (X_n) IF THE OCCURANCE OR NONOCCURANCE OF THE EVENT "WE STOP AT TIME n" $(T=n)$ CAN BE DETERMINED BY LOOKING AT $X_0, X_1, \ldots, X_n$.

CONSTANT BETTING UP TO A STOPPING TIME.

ONE POSSIBLE BETTING STRATEGY IS TO BET \$1 ON EACH GAME UNTIL SOME TIME T AT WHICH TIME WE STOP. THAT IS,

$$H_m = \begin{cases} 1 & \text{IF } m < T \\ 0 & \text{\_\_\_\_\_}. \end{cases}$$

NOTICE THAT T IS A STOPPING TIME, AND THUS H_m IS A FUNCTION OF $X_0, X_1, \ldots, X_{m-1}$.

NOW, $W_n = Y_0 + \sum_{m=1}^{n} H_m (Y_m - Y_{m-1}) = Y_{\min\{T,n\}}$

(THE BOOK USES THE NOTATION $T \wedge n = \min\{T, n\}$).

IT FOLLOWS THAT $W_n = Y_{T\wedge n}$ IS A SUPERMARTINGALE

<u>THM</u>: IF (Y_n) IS A SUPERMARTINGALE (MARTINGALE, SUBMARTINGALE) WITH RESPECT TO (X_n) AND T IS A STOPPING TIME WITH RESPECT TO (X_n), THEN THE STOPPED PROCESS $Y_{T\wedge n}$ IS A SUPERMARTINGALE (MARTINGALE, SUBMARTINGALE) W.R.T. (X_n).

LET $\xi_1, \xi_2, \ldots$ BE INDEPENDENT AND HAVE

$$P\{\xi_i = 1\} = 1/2, \quad P\{\xi_i = -1\} = 1/2$$

AND

LET $S_n = \xi_1 + \ldots + \xi_n$

LET $T = \min\{n \geq 1 : S_n = 1\}$

$P\{T_1 < \infty\} = 1$ WHY? (RANDOM WALK EXAMPLE IMPLIES THAT 1 IS NULL-RECURRENT)

IT FOLLOWS THAT WITH PROBABILITY 1, AT SOME POINT IN TIME WE WILL HAVE \$1 IN PROFIT.

THM: (STOPPING THM FOR BOUNDED MARTINGALES).

LET M_n BE A MARTINGALE WITH RESPECT TO X_n AND LET T BE A STOPPING TIME WITH RESPECT TO X_n WITH

$$P\{T < \infty\} = 1$$

IF $\exists K \ni |M_{T \wedge n}| \leq K$ FOR ALL n, THEN

$$E[M_T] = E[M_0].$$

Pf:

$$E[M_{T\wedge n} - M_T]$$
$$= E[M_{T\wedge n} - M_T \mid T > n]\, P\{T > n\}$$
$$+ E[M_{T\wedge n} - M_T \mid T \leq n]\, P\{T \leq n\}$$

NOW, $$M_{T\wedge n} - M_T = M_0 + \sum_{m=1}^{n} 1_{T>m}(M_m - M_{m-1}) - M_T$$

AND IF $T > n$ THEN $1 \leq m \leq n < T$ IN WHICH CASE

$$M_{T\wedge n} - M_T = M_0 + \sum_{m=1}^{n}(M_m - M_{m-1}) - M_T$$
$$= M_n - M_T$$

ON THE OTHERHAND, IF $T \leq n$ THEN

$$M_{T\wedge n} - M_T = M_0 + \sum_{m=1}^{T}(M_m - M_{m-1}) - M_T = 0$$

THUS

$$E[M_{T\wedge n} - M_T] = E[M_n - M_T]\, P\{T > n\}$$

AND THUS

$$E[M_{T\wedge n} - M_T] = \left(E[M_n] - E[M_T]\right) P\{T > n\}$$

IN WHICH CASE

$$\left| E[M_{T\wedge n}] - E[M_T] \right| \leq 2K \, P\{T > n\} \longrightarrow 0$$

AS $n \to \infty$

THUS, SINCE $E[M_{T\wedge n}] = E[M_0]$ WE HAVE

$\left| E[M_0] - E[M_T] \right| = 0$ WHICH PROVES THE THEOREM. ✓

LET $S_n = S_0 + \xi_1 + \ldots + \xi_n$ WHERE

$$P(\xi_i = 1) = P(\xi_i = -1) = 1/2$$

THIS IS YOUR POSITION AT STEP n OF A RANDOM WALK STARTING AT S_0.

(S_n) IS A MARTINGALE.

LET $T = \min\{n \mid S_n \in \{a, b\}\}$. THAT IS, THE FIRST TIME THAT THE WALKER HITS EITHER a OR b.

T IS CLEARLY A STOPPING TIME.

THE PEDESTRIAN LEMMA IMPLIES THAT $P\{T < \infty\} = 1$.

NOW, IF THE WALK STARTS AT $S_0 = x$

$$x = E[S_0] = E[S_T] = a\,P\{S_T = a\} + b\,P\{S_T = b\}$$

SINCE $P\{S_T = a\} = 1 - P\{S_T = b\}$, IMPLIES

$$x = a + (b-a)\,P\{S_T = b\}.$$

INTRODUCING $V_y = \min\{n \geq 0 : X_n = y\}$ WE HAVE

$$P\{S_T = b \mid S_0 = x\} = P_x\{V_b < V_a\} = \frac{x-a}{b-a}.$$

GAMBLER'S RUIN

NOW LET $S_n = S_0 + \xi_1 + \dots + \xi_n$ WHERE

$$P\{\xi_i = 1\} = p, \quad P\{\xi_i = -1\} = q = 1-p$$

AGAIN, LET $T = \min\{n \mid S_n \in \{a,b\}\}$

SETTING $g(x) = \left(\frac{q}{p}\right)^x$ THEN $(g(S_n))$ IS A MARTINGALE.

T IS CLEARLY A STOPPING TIME AND $P\{T<\infty\}=1$.
IT FOLLOWS THAT IF $S_0 = x$

$$\left(\frac{q}{p}\right)^x = E[g(S_0)] = \left(\frac{q}{p}\right)^a P\{S_T = a\} + \left(\frac{q}{p}\right)^b P\{S_T = b\}$$

SINCE $P\{S_T = a\} = 1 - P\{S_T = b\}$, THUS

$$\left(\frac{q}{p}\right)^x = \left(\frac{q}{p}\right)^a + \left\{\left(\frac{q}{p}\right)^b - \left(\frac{q}{p}\right)^a\right\} P\{S_T = b\}$$

AND SOLVING FOR $P\{S_T = b\}$ WE HAVE

$$P_x\{V_b < V_a\} = \frac{(q/p)^x - (q/p)^a}{(q/p)^b - (q/p)^a}.$$

HW: 2.5.16, 2.5.19

CHAPTER 3: POISSON PROCESSES

<u>DEF</u>: A RANDOM VARIABLE T IS SAID TO BE <u>EXPONENTIALLY DISTRIBUTED</u> WITH PARAMETER (RATE) λ IF

$$P\{T \le t\} = 1 - e^{-\lambda t} \quad \forall\, t \ge 0$$

<u>NOTATION</u>: T = EXPONENTIAL(λ).

OF COURSE IF $F_T(t) = P\{T \le t\} = 1 - e^{-\lambda t}$ THEN

$$f_T(t) = \frac{d}{dt} F_T(t) = \begin{cases} \lambda e^{-\lambda t} & \text{IF } t \ge 0 \\ 0 & \text{———} \end{cases}$$

THE FOLLOWING IS EASY TO VERIFY

PROP: i) $E[T] = \frac{1}{\lambda}$

ii) $\mathrm{VAR}(T) = \frac{1}{\lambda^2}$

iii) IF $S = $ EXPONENTIAL(1), THEN $T = \frac{S}{\lambda}$

MEMORYLESSNESS.

IF T IS AN EXPONENTIAL RANDOM VARIABLE WITH RATE λ THEN

$$P\{T > t+s \mid T > t\} = P\{T > s\}$$

Pf.

$$P\{T > t+s \mid T > t\} = \frac{P\{T > t+s\}}{P\{T > t\}} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}}$$

$$= e^{-\lambda s} = P\{T > s\}.$$

EXPONENTIAL RACES

SUPPOSE $S = $ EXPONENTIAL(λ) AND $T = $ EXPONENTIAL(μ) BE INDEPENDENT. NOTE THEN

$$P\{S \wedge T > t\} = P\{S > t, T > t\}$$
$$= P\{S > t\} P\{T > t\} = e^{-\lambda t} e^{-\mu t} = e^{-(\lambda+\mu)t}$$

SIMILARLY IF FOR $i = 1, 2, \ldots, n$, (T_i) ARE INDEPENDENT WITH

$T_i = $ EXPONENTIAL(λ_i)

$$P\{\min\{T_1, \ldots, T_n\} > t\} = e^{-(\lambda_1 + \lambda_2 + \ldots + \lambda_n)t}$$

RETURNING TO THE CASE OF JUST TWO RACERS, WE MAY ASK:

WHO FINISHES FIRST?

WE NOTE THAT

$$P\{S < T\} = \int_0^\infty \int_S^\infty \mu e^{-\mu t} \lambda e^{-\lambda s} \, dt \, ds$$

$$= \int_0^\infty \lambda e^{-\lambda s} \int_s^\infty \mu e^{-\mu t} \, dt \, ds$$

$$= \int_0^\infty \lambda e^{-\lambda s} \left[-e^{-\mu t} \right|_s^\infty dt = \int_0^\infty \lambda e^{-(\lambda+\mu)s} \, ds$$

$$= \left[-\frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)s} \right|_0^\infty = \frac{\lambda}{\lambda+\mu}$$

SIMILARLY, ONE CAN SHOW THAT

$$P\{T_i = \min\{T_1, \ldots, T_n\}\} = \frac{\lambda_i}{\lambda_1 + \ldots + \lambda_n}$$

THUS, IF WE DENOTE BY I THE INDEX FOR WHICH

$$T_I = \min\{T_1, \ldots, T_n\}$$

I IS A RANDOM VARIABLE WITH

$$P\{I = i\} = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \ldots + \lambda_n}$$

LET $V = \min\{T_1, \ldots, T_n\}$ WHICH IS THE TIME OF THE WINNING RACER.

<u>CLAIM</u>: I AND V ARE INDEPENDENT.

<u>Pf</u>: $P\{I = i, V = t\} = P\{T_i = t, T_j > t \text{ FOR } j \neq i\}$

$$= \underbrace{\lambda_i e^{-\lambda_i t}}_{\text{PROBABILITY DENSITY}} \prod_{j \neq i} e^{-\lambda_j t}$$

$$= \frac{\lambda_i}{\lambda_1 + \lambda_2 + \ldots + \lambda_n} \cdot (\lambda_1 + \ldots + \lambda_n) e^{-(\lambda_1 + \ldots + \lambda_n)t}$$

$$= P\{I = i\} P\{V = t\} \quad \checkmark$$

DEF: A RANDOM VARIABLE Γ IS SAID TO BE GAMMA DISTRIBUTED WITH PARAMETERS n AN λ IF

$$P\{\Gamma = \gamma\} = \lambda e^{-\lambda\gamma} \frac{(\lambda\gamma)^{n-1}}{(n-1)!} \quad \text{FOR } \gamma \geq 0.$$

NOTATION: $\Gamma = \text{gamma}(n, \lambda)$.

THM: SUPPOSE $T_1, T_2, \ldots, T_n$ ARE INDEPENDENT EXPONENTIAL RANDOM VARIABLES WITH PARAMETERS ALL EQUAL TO λ. THEN

$$\Gamma_n = T_1 + T_2 + \ldots + T_n$$

IS $\text{gamma}(n, \lambda)$.

Pf: IF $n = 1$ THEN

$$P\{T_1 = \gamma\} = \lambda e^{-\lambda\gamma} = \lambda e^{-\lambda\gamma} \frac{(\lambda\gamma)^{n-1}}{(n-1)!}$$

$$= P\{\Gamma = \gamma\}$$

NOW ASSUME

$$P\{\Gamma_n = \gamma\} = \lambda e^{-\lambda\gamma} \frac{(\lambda\gamma)^{n-1}}{(n-1)!}$$

AND LOOK AT

$$P\{\Gamma_{n+1} = t\} = \int_0^t P\{\Gamma_n = s\} P\{T_{n+1} = t-s\} \, ds$$

$$= \int_0^t \lambda e^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!} \lambda e^{-\lambda(t-s)} \, ds$$

$$= e^{-\lambda t} \lambda^{n+1} \int_0^t \frac{s^{n-1}}{(n-1)!} \, ds = e^{-\lambda t} \lambda^{n+1} \frac{t^n}{n!}$$

$$= \lambda e^{-\lambda t} \frac{(\lambda t)^n}{n!} \quad \text{AS DESIRED}$$

HW. 3.7.1, 3.7.2, 3.7.3, 3.7.5

§ 3.2 : DEFINING THE POISSON PROCESS

DEF: LET $t_1, t_2, \ldots$ BE INDEPENDENT EXPONENTIAL RANDOM VARIABLES. ALL WITH RATE λ. LET

$$T_n = t_1 + \ldots + t_n \quad \text{FOR } n \geq 1, \; T_0 = 0$$

AND DEFINE $N(s) = \max\{n : T_n \leq s\}$

WE THINK OF THE t_n AS TIMES BETWEEN ARRIVALS OF CUSTOMERS AT A BANK SO THAT $T_n = t_1 + \ldots + t_n$ IS THE TIME OF ARRIVAL OF THE nTH CUSTOMER AND $N(s)$ IS THE NUMBER OF ARRIVALS BY TIME s

LET US COMPUTE THE DISTRIBUTION OF $N(s)$. $N(s) = n$ IF

$$T_n \leq s < T_{n+1}$$

THIS MEANS THAT IF $T_n = t$, $T_{n+1} - T_n = t_{n+1} > s - t$ AND t_{n+1} IS INDEPENDENT OF T_n. THUS,

$$
\begin{aligned}
P\{N(s) = n\} &= \int_0^s P\{T_n = t\}\, P\{T_{n+1} > s \mid T_n = t\}\, dt \\
&= \int_0^s P\{T_n = t\}\, P\{t_{n+1} > s - t\}\, dt \\
&= \int_0^s \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda(s-t)}\, dt \\
&= \frac{\lambda^n}{(n-1)!} e^{-\lambda s} \int_0^s t^{n-1}\, dt = e^{-\lambda s} \frac{(\lambda s)^n}{n!}
\end{aligned}
$$

RECALL THAT X HAS A POISSON DISTRIBUTION WITH MEAN μ, OR $X = \text{POISSON}(\mu)$ IF

$$P\{X = n\} = e^{-\mu} \frac{\mu^n}{n!} \quad \text{FOR } n = 0, 1, 2, \ldots$$

IT FOLLOWS THAT

$N(s)$ IS A POISSON RANDOM VARIABLE WITH MEAN λs.

THIS EXPLAINS OUR USING THE WORD RATE FOR λ.

EX. 3.2.1

WE CALL $\{N(s); s > 0\}$ A POISSON PROCESS.

AN ALTERNATE DEFINITION OF POISSON PROCESSES:

LEMMA: $N(t+s) - N(s)$, $t \geq 0$, IS A RATE λ POISSON PROCESS AND IS INDEPENDENT OF $N(r)$, $0 \leq r \leq s$

Pf: SUPPOSE BY THE TIME s THERE HAVE BEEN n ARRIVALS $T_1, T_2, \ldots, T_n$ WHICH OCCURRED AT $u_1, u_2, \ldots, u_n$ THE $(n+1)$ST ARRIVAL MUST HAVE $t_{n+1} > s - u_4$, BUT THEN

$$P\{t_{n+1} > s - u_4 + t \mid t_{n+1} > s - u_4\}$$
$$= P\{t_{n+1} > t\} = e^{-\lambda t}$$

BY THE MEMORYLESSNESS OF t_{n+1}.

IT FOLLOWS THAT THE FIRST ARRIVAL AFTER s IS EXPONENTIAL(λ). AND INDEPENDENT OF $T_1, T_2, \ldots, T_n$.

BUT THEN, ALL ARRIVAL TIMES AFTER s ARE INDEPENDENT EXPONENTIAL(λ) RANDOM VARIABLES, AND THUS

$N(t+s) - N(s)$, $t \geq 0$ IS A POISSON PROCESS.

LEMMA: $N(s)$ HAS INDEPENDENT INCREMENTS: IF $s_0 < s_1 < \ldots < s_n$ THEN

$$N(s_1) - N(s_0), N(s_2) - N(s_1), \ldots, N(s_n) - N(s_{n-1})$$

ARE INDEPENDENT.

Pf: THE PREVIOUS LEMMA IMPLIES THAT $N(s_n) - N(s_{n-1})$ IS INDEPENDENT OF $N(r)$ $\forall r \leq s_{n-1}$, AND THUS OF

$$N(s_m) - N(s_{m-1}) \quad \text{IF} \quad m < n.$$

THE LEMMA FOLLOWS EASILY BY INDUCTION.

THM: IF $\{N(s) : s > 0\}$ IS A POISSON PROCESS, THEN

i) $N(0) = 0$

ii) $N(t+s) - N(s) =$ POISSON(λt)

iii) $N(t)$ HAS INDEPENDENT INCREMENTS.

CONVERSELY IF $\{N(s) : s > 0\}$ IS SUCH THAT i, ii, iii HOLD, THEN IT IS A POISSON POINT PROCESS.

Pf: WE HAVE PROVED ALL BUT THE CONVERSE.

IF THERE ARE NO ARRIVALS IN $[0,t]$, THEN $T_1 > t$ AND

$$P\{T_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$$

SINCE $N(t+0) - N(0) = N(t)$ IS POISSON(λ).

NOW, FOR $t_2 = T_2 - T_1$,

$$P\{t_2 > t \mid t_1 = s\} = P\{\text{NO ARRIVALS IN } (s, s+t] \mid t_1 = s\}$$
$$= P\{N(t+s) - N(s) = 0 \mid N(r) = 0 \text{ FOR } r < s, N(s) = 1\}$$
$$= P\{N(t+s) - N(s) = 0\} = e^{-\lambda t}$$

BY THE INDEPENDENT INCREMENT PROPERTY.

REPEATING FOR $t_3, t_4, \ldots$ WE SEE THAT $t_1, t_2, \ldots, t_n$ ARE INDEPENDENT EXPONENTIAL(λ). ✓

WE WILL TAKE THIS THM TO BE OUR SECOND DEFINITION OF A POISSON PROCESS.

WHY IS THE POISSON PROCESS SO IMPORTANT?

SUPPOSE WE HAVE n INDIVIDUALS EACH OF WHICH DECIDES TO GO TO THE BANK INDEPENDENTLY ON ANY PARTICULAR DAY WITH PROBABILITY λ/n, AND SUPPOSE THAT A PERSON WHO DECIDES TO GO TO THE BANK WILL DO SO AT A TIME CHOSEN AT RANDOM UNIFORMLY FROM THE BANKS HOURS OF BUSINESS. NOW, THE PROBABILITY THAT EXACTLY k PEOPLE WILL GO TO THE BANK IS GIVEN BY THE BINOMIAL DISTRIBUTION

$$\frac{n(n-1)\cdot\ldots\cdot(n-k+1)}{k!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{\lambda^k}{k!}\,\frac{n(n-1)\cdot\ldots\cdot(n-k+1)}{n^k}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}$$

AND THUS, IF n IS LARGE (OR MORE PRECISELY, TAKING $n \to \infty$) WE HAVE

$$\longrightarrow \frac{\lambda^k}{k!}e^{-\lambda}$$

AND THUS, IN THE LIMIT, THE NUMBER OF PEOPLE GOING TO THE BANK ON A GIVEN DAY IS A POISSON RANDOM VARIABLE.

NOW, LET US ASK WHAT IS THE NUMBER OF INDIVIDUALS WHO GO TO THE BANK IN THE FIRST, SAY 1/4TH, OF THE DAY.

SAY j GO IN THE FIRST FOURTH AND k GO IN THE REMAINING THREE FOURTHS OF THE DAY. THE MULTINOMIAL DISTRIBUTION IMPLIES THAT THE PROBABILITY OF THIS OCCURRING IS

$$\frac{n!}{j!\,k!\,(n-j-k)!}\left(\frac{1}{4}\frac{\lambda}{n}\right)^{j}\left(\frac{3}{4}\frac{\lambda}{n}\right)^{k}\left(1-\frac{\lambda}{n}\right)^{n-(j+k)}$$

$$= \frac{(\lambda/4)^j}{j!}\frac{(3\lambda/4)^k}{k!}\frac{n(n-1)\cdots(n-j-k+1)}{n^{j+k}}\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-(j+k)}$$

$$\longrightarrow \frac{(\lambda/4)^j}{j!}\frac{(3\lambda/4)^k}{k!}e^{-\lambda} \quad \text{AS } n \to \infty.$$

$$= \frac{(\lambda/4)^j}{j!}e^{-\lambda/4}\,\frac{(3\lambda/4)^k}{k!}e^{-3\lambda/4}$$

THIS SHOWS THAT THE NUMBER OF ARRIVALS IN THE FIRST FOURTH, AND THE REMAINING THREE FOURTHS OF THE DAY ARE INDEPENDENT POISSONS WITH PARAMETERS $\lambda/4$ AND $3\lambda/4$.

THIS ARGUMENT CAN BE EXTENDED FOR ANY NUMBER OF SUBDIVISIONS OF THE DAY

THERE ARE SOME WEAKNESSES OF THE PRECEEDING ARGUMENT

i) ALL INDIVIDUALS IN THE POPULATION HAVE THE SAME PROBABILITY OF GOING TO THE BANK ON A GIVEN DAY.

ii) INDIVIDUALS WHO DO GO, DO SO UNIFORMLY THROUGHOUT THE DAY

THE FIRST ASSUMPTION CAN BE WEAKENED.

THM: FOR EACH n, LET $X_{n,m}$ BE INDEPENDENT BERNOULLI RANDOM VARIABLES WITH

$$P\{X_{n,m}=1\} = p_{n,m} \qquad P\{X_{n,m}=0\} = 1-p_{n,m}$$

LET

$$S_n = X_{n,1} + \cdots + X_{n,n}$$
$$\lambda_n = E[S_n] = p_{n,1} + \cdots + p_{n,n}$$

AND SET

$$Z_n = \text{POISSON}(\lambda_n).$$

THEN, FOR ANY SET A

$$\left| P\{S_n \in A\} - P\{Z_n \in A\} \right| \leq \sum_{m=1}^{n} p_{n,m}^2$$

WE WILL NOT PROVE THIS, BUT IT IMPLIES THE FOLLOWING NICE COROLLARY:

COR: SUPPOSE $\lambda_n \to \lambda < \infty$ AND

$$\max_k p_{n,k} \to 0$$

THEN

$$\max_A \left| P\{S_n \in A\} - P\{Z_n \in A\} \right| \to 0.$$

THAT IS, IF n IS LARGE, AND EACH OF THE $p_{n,k}$ ARE SMALL, THEN S_n IS "ALMOST" POISSON WITH MEAN

$$\lambda_n = p_{n,1} + \cdots + p_{n,n}.$$

TO ADDRESS PROBLEM ii), LET US GENERALIZE THE DEFINITION OF A POISSON PROCESS AS FOLLOWS:

DEF: (NONHOMOGENEOUS POISSON PROCESS). WE SAY $\{N(s), s \geq 0\}$ IS A POISSON PROCESS WITH RATE $\lambda(r)$, IF:

i) $N(0)=0$

ii) $N(t)$ HAS INDEPENDENT INCREMENTS

iii) $N(t+s) - N(s)$ IS POISSON WITH MEAN

$$\int_s^t \lambda(r)\, dr.$$

THE FOLLOWING SHOWS THAT THE FIRST DEFINITION OF POISSON PROCESS DOESN'T GENERALIZE SO EASILY TO THIS SITUATION SINCE

$$P\{t_1 > t\} = P\{N(t) = 0\} = e^{-\int_0^t \lambda(s)\,ds}$$

AND DIFFERENTIATING GIVES

$$P\{t_1 = t\} = -\frac{d}{dt} P\{t_1 > t\} = \lambda(t) \exp\left\{-\int_0^t \lambda(s)\,ds\right\}$$

WHICH IS NOT, IN GENERAL AN EXPONENTIAL RANDOM VARIABLE (UNLESS $\lambda(r)$ IS A CONSTANT).

A SIMILAR CALCULATION SHOWS THAT

$$P\{t_1 = s, t_2 = t\} = \lambda(s) \exp\left\{-\int_0^s \lambda(r)\,dr\right\}$$
$$\times \lambda(s+t) \exp\left\{-\int_0^{s+t} \lambda(r)\,dr - \int_0^s \lambda(r)\,dr\right\}$$

WHICH IMPLIES THAT t_1 AND t_2 ARE NOT INDEPENDENT WHEN $\lambda(r)$ IS NOT CONSTANT. (WHY?).

WE MAY THEREFORE USE THIS GENERALIZATION TO DEFINE A NON-HOMOGENEOUS POISSON PROCESS WHERE THE TIME A PERSON GOES TO THE BANK IS NOT UNIFORM.

EX. 3.2.2, 3.7.16, 3.7.17

§3.3: COMPOUND POISSON PROCESSES

IN THIS SECTION WE WILL ATTACH TO EACH ARRIVAL TIME AN i.i.d. RANDOM VARIABLE Y_i.

SUPPOSE $\{N(s) : s \geq 0\}$ IS A POISSON PROCESS WHICH REPRESENTS THE NUMBER OF CARS THAT HAVE DRIVEN THROUGH A DRIVE-THROUGH UP TO A CERTAIN TIME.

IF t_i IS AN ARRIVAL TIME, LET Y_i BE THE NUMBER OF PEOPLE IN THE CAR.

DEFINE $S(t) = \sum_{i:\, t_i < t} Y_i$

IN THE PREVIOUS EXAMPLE $S(t)$ IS THE NUMBER OF PEOPLE WHO HAVE GONE THROUGH THE DRIVE-THROUGH UP TO TIME t.

THM: LET $Y_1, Y_2, \ldots$ BE i.i.d. RANDOM VARIABLES AND LET N BE A NON-NEGATIVE INTEGER VALUED RANDOM VARIABLE AND SET

$$S = Y_1 + \ldots + Y_N \quad \text{WITH } S = 0 \text{ WHEN } N = 0$$

i) IF $E[N] < \infty$ THEN $E[S] = E[N]\,E[Y_i]$

ii) IF $E[N^2] < \infty$ THEN

$$\text{var}(S) = E[N]\,\text{var}(Y_i) + \text{var}(N)\,E[Y_i]^2$$

iii) IF $N = \text{Poisson}(\lambda)$ THEN

$$\text{var}(S) = \lambda\,E[Y_i^2].$$

Pf: WHEN $N = n$, $E[S] = n\,E[Y_i]$, THUS

$$E[S] = \sum_{n=0}^{\infty} E[S \mid N = n]\,P\{N = n\}$$

$$= \sum_{n=0}^{\infty} n\,E[Y_i]\,P\{N=n\} = E[Y_i] \sum_{n=0}^{\infty} n\,P\{N=n\}$$

$$= E[N]\,E[Y_i]$$

AGAIN, WHEN $N = n$, $\text{var}(S) = n\,\text{var}(Y_i)$, AND

$$E[S^2 \mid N = n] = n\,\text{var}(Y_i) + (n\,E[Y_i])^2$$

IT FOLLOWS THAT

$$E[S^2] = \sum_{n=0}^{\infty} E[S^2 \mid N = n]\,P\{N = n\}$$

$$= \text{var}(Y_i) \sum_{n=0}^{\infty} n\,P\{N=n\} + E[Y_i]^2 \sum_{n=0}^{\infty} n^2\,P\{N=n\}$$

$$= \text{var}(Y_i)\,E[N] + E[Y_i]^2\,E[N^2]$$

THUS,

$$\begin{aligned} \text{var}(S) &= E[S^2] - E[S]^2 \\ &= \text{var}(Y_i)E[N] + E[Y_i]^2 E[N^2] - (E[N]E[Y_i])^2 \\ &= \text{var}(Y_i)E[N] + E[Y_i]^2 (E[N^2] - E[N]^2) \\ &= E[N]\,\text{var}(Y_i) + \text{var}(N)\,E[Y_i]^2 \end{aligned}$$

NOW, IF $N = \text{POISSON}(\lambda)$, THEN

$$E[N] = \text{var}(N) = \lambda,$$ AND THUS

$$\begin{aligned} \text{var}(S) &= \lambda\,\text{var}(Y_i) + \lambda E[Y_i]^2 \\ &= \lambda E[Y_i^2] \end{aligned}$$ AS EXPECTED. ✓

SUPPOSE THE NUMBER OF CUSTOMERS AT A LIQUOR STORE IN A DAY HAS A POISSON DISTRIBUTION WITH MEAN 81, AND THAT EACH CUSTOMER SPENDS AN AVERAGE OF \$8 WITH A STANDARD DEVIATION OF \$6.

THE MEAN REVENUE FOR THE DAY IS THUS

$$(81)(6) = \$648.$$

THE VARIANCE OF THE TOTAL REVENUE IS

$$81(6^2 + 8^2) = 8100$$

THAT IS THE STANDARD DEVIATION IS \$90.

HW: 3.7.29, 3.7.30

§ 3.4: THINNING AND SUPERPOSITION

IN SECTION 3.3 WE USED THE Y_i TO SEE HOW MUCH OF A QUANTITY (CUSTOMERS, MONEY, ETC.) ACCUMULATED BY A CERTAIN TIME GIVEN THAT THE ARRIVALS WERE POISSON.

IN THIS SECTION WE WILL USE THE Y_i TO SPLIT OUR POISSON PROCESS INTO SEVERAL.

FOR INSTANCE, LET

$$N_j(t) = \#\{i \le N(t) : y_i = j\}$$

FOR INSTANCE, IF $N(t)$ IS THE NUMBER OF CARS THROUGH A DRIVE-THROUGH UP TO TIME t, AND y_i IS THE NUMBER OF PEOPLE IN EACH CAR; $N_3(t)$ IS THE NUMBER OF CARS THROUGH THE DRIVE THROUGH WITH 3-PEOPLE IN THEM UP TO TIME t. THIS SPLITS $N(t)$ INTO THE PROCESSES, $N_1(t), N_2(t), \ldots$

<u>THM</u>: $N_j(t)$ IS A POISSON PROCESS WITH RATE $\lambda P\{y_i = j\}$. IF $j \ne j'$ THEN $N_j(t)$ AND $N_{j'}(t)$ ARE INDEPENDENT.

IDEA OF PROOF:

FOR SIMPLICITY SUPPOSE $P\{y_i = 1\} = p$, $P\{y_i = 2\} = 1-p$

BY THE INDEPENDENT INCREMENTS OF $N(t)$, IT FOLLOWS THAT PAIRS

$$\left(N_1(s_i) - N_1(s_{i-1}),\ N_2(s_i) - N_2(s_{i-1})\right) \quad 1 \le i \le n$$

ARE INDEPENDENT OF EACH OTHER. IN PARTICULAR

$N_1(s_i) - N_1(s_{i-1})$ ARE INDEPENDENT OF EACH OTHER

AS ARE

$N_2(s_i) - N_2(s_{i-1})$.

CLEARLY $N(0) = 0 \Rightarrow N_1(0) = 0$ AND $N_2(0) = 0$.

IT REMAINS ONLY TO SHOW THAT

$$Y = N_1(t+s) - N_1(s) \quad \text{AND} \quad Z = N_2(t+s) - N_2(s)$$

ARE INDEPENDENT AND POISSON.

NOW, IF $Y = j$ AND $Z = k$, THEN $N(t+s) - N(s) = j+k$, AND

$$P\{Y = j, Z = k\} = e^{-\lambda t}\frac{(\lambda t)^{j+k}}{(j+k)!}\frac{(j+k)!}{j!\,k!}p^j(1-p)^k$$

$$= e^{-\lambda p t}\frac{(\lambda p t)^j}{j!}e^{-\lambda(1-p)t}\frac{(\lambda(1-p)t)^k}{k!}$$

WHICH IS THE DESIRED RESULT. ✓

<u>THM</u>: SUPPOSE $N_1(t), \ldots, N_k(t)$ ARE INDEPENDENT POISSON PROCESSES WITH RATES $\lambda_1, \ldots, \lambda_k$. THEN

$$N_1(t) + \ldots + N_k(t) = N(t)$$

IS A POISSON PROCESS WITH RATE $\lambda_1 + \ldots + \lambda_k$.

<u>Pf</u>: CLEARLY $N(0) = 0$ AND $N(t)$ HAS INDEPENDENT INCREMENTS. FINALLY, SINCE THE SUM OF INDEPENDENT POISSON'S WITH PARAMETERS λ AND μ IS A POISSON WITH PARAMETER $\lambda+\mu$, THE THEOREM FOLLOWS. ✓

A POISSON RACE.

GIVEN A POISSON PROCESS OF RED ARRIVALS WITH RATE λ AND A POISSON PROCESS OF GREEN ARRIVALS WITH RATE μ

WHAT IS THE PROBABILITY OF 6 RED ARRIVALS BEFORE 4 GREEN ONES?

LET $R(t)$ AND $G(t)$ BE THE NUMBER OF RED AND GREEN ARRIVALS UP TO TIME t, AND SET

$$N(t) = R(t) + G(t)$$

$$P\{R(t) = 6 \text{ BEFORE } G(t) = 4\}$$
$$= P\{N(t) = 9 \text{ AND } R(t) \geq 6\}$$

NOW, $p = \dfrac{\lambda}{\lambda+\mu}$ THEN

$$P\{N(t) = 9 \text{ AND } R(t) \geq 6\} = \sum_{k=6}^{9} \binom{9}{k} p^k (1-p)^{9-k}.$$

HW: 3.4.1, 1.7.12, 1.7.24, 1.7.32

§ 3.5 CONDITIONING

SUPPOSE $T_1, T_2, \ldots$ ARE THE ARRIVAL TIMES OF A POISSON PROCESS WITH RATE λ, AND LET $U_1, U_2, \ldots$ BE INDEPENDENT AND UNIFORMLY DISTRIBUTED ON $[0, t]$.

<u>THM</u>: IF WE CONDITION ON $N(t)=n$, THEN THE SET OF ARRIVAL TIMES $\{T_1, \dots, T_n\}$ HAS THE SAME DISTRIBUTION AS $\{U_1, \dots, U_n\}$.

IDEA OF PROOF:

LET'S LOOK AT

$$P\{T_1 = v_1, T_2 = v_2, T_3 = v_3 \mid N(4) = 3\}$$

NOTICE THAT THIS PROBABILITY IS 0 UNLESS $0 < v_1 < v_2 < v_3 < 4$. NOW,

$$P\{T_1 = v_1, T_2 = v_2, T_3 = v_3 \mid N(4) = 3\}$$

$$= \frac{P\{t_1 = v_1, t_2 = v_2 - v_1, t_3 = v_3 - v_2, t_4 > 4 - v_3\}}{P\{N(4) = 3\}}$$

$$= \frac{\lambda e^{-\lambda v_1} \lambda e^{-\lambda(v_2 - v_1)} \lambda e^{-\lambda(v_3 - v_2)} e^{-\lambda(4 - v_3)}}{e^{-4\lambda} 4^3/3!}$$

$$= \frac{\lambda^3 e^{-4\lambda}}{e^{-4\lambda} 4^3/3!} = \frac{3!}{4^3}$$

THIS IMPLIES THAT THE DISTRIBUTION IS UNIFORM OVER

$$\{(v_1, v_2, v_3) : 0 < v_1 < v_2 < v_3 < 4\}$$

SINCE $\operatorname{vol}\{(v_1, v_2, v_3) : 0 < v_1, v_2, v_3 < 4\} = 4^3$

A SIMILAR ARGUMENT SHOWS

$$P\{T_1 = v_1, \dots, T_n = v_n \mid N(t) = n\} = \frac{n!}{t^n}$$

NOW, GIVEN $U_1, \dots, U_n$, LET $U_{(1)}, U_{(2)}, \dots, U_{(n)}$ BE THE REORDERING SUCH THAT

$$U_{(1)} < U_{(2)} < \dots < U_{(n)}$$

$U_{(m)}$ IS CALLED THE m^{TH} ORDER STATISTIC OF $U_1, \dots, U_n$

SINCE

$$P\{U_1 = u_1, \dots, U_n = u_n\} = \frac{1}{t^n} \quad \text{FOR} \quad 0 < u_1, \dots, u_n < t$$

AND SINCE SORTING MAPS $n!$ VECTORS TO 1,

$$P\{U_{(1)} = u_1, \dots, U_{(n)} = u_n\} = \frac{n!}{t^n} \quad \text{FOR} \quad 0 < u_1 < \dots < u_n < t$$

IT FOLLOWS THAT

$(T_1, \dots, T_n)$ AND $(U_{(1)}, \dots, U_{(n)})$ HAVE THE SAME DISTRIBUTION.

THUS

$\{T_1, \dots, T_n\}$ AND $\{U_1, \dots, U_n\}$ HAVE THE SAME DISTRIBUTION. ✓

<u>THM</u>: IF $s < t$ AND $0 \le m \le n$, THEN

$$P\{N(s) = m \mid N(t) = n\} = \binom{n}{m} \left(\frac{s}{t}\right)^m \left(1 - \frac{s}{t}\right)^{n-m}$$

<u>pf</u>:

$$P\{N(s) = m \mid N(t) = n\} = \frac{P\{N(s) = m, N(t) - N(s) = n - m\}}{P\{N(t) = n\}}$$

$$= \frac{P\{N(s) = m\} P\{N(t) - N(s) = n - m\}}{P\{N(t) = n\}}$$

$$= e^{-\lambda s} \frac{(\lambda s)^m}{m!} e^{-\lambda(t-s)} \frac{(\lambda (t-s))^{n-m}}{(n-m)!} \Big/ e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

$$= \binom{n}{m} \left(\frac{s}{t}\right)^m \left(1 - \frac{s}{t}\right)^{n-m}. \checkmark$$

HW: 3.7.

CHAPTER 6: BROWNIAN MOTION

<u>DEF</u>: $\{B(t) : t \in [0,\infty)\}$ IS A (ONE-DIMENSIONAL) BROWNIAN MOTION WITH VARIANCE σ^2 IF IT SATISFIES THE FOLLOWING CONDITIONS:

a) $B(0) = 0$

b) INDEPENDENT INCREMENTS: IF $0 = t_0 < t_1 < \ldots < t_n$ THEN

$$B(t_1) - B(t_0), \ldots, B(t_n) - B(t_{n-1})$$

ARE INDEPENDENT

c) STATIONARY INCREMENTS:

$B(t) - B(s)$ DEPENDS ONLY ON $t - s$

d) $B(t) = N(0, \sigma^2 t)$

e) AS A FUNCTION, $B(t)$ IS CONTINUOUS.

RECALL THE CENTRAL LIMIT THEOREM:

LET $X_1, X_2, \ldots,$ BE i.i.d. WITH $E[X_i] = 0$ AND $\text{VAR}(X_i) = \sigma^2$, AND LET $S_n = X_1 + \ldots + X_n$

THEN FOR ALL x, WE HAVE

$$P\left\{\frac{S_n}{\sigma\sqrt{n}} \le x\right\} \longrightarrow P\{\chi \le x\}$$

WHERE $\chi = N(0,1)$. THAT IS

$$P\{\chi \le x\} = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy.$$

FORMALLY, WE SAY $\frac{S_n}{\sigma\sqrt{n}}$ <u>CONVERGES IN DISTRIBUTION</u> TO χ AND WRITE

$$\frac{S_n}{\sigma\sqrt{n}} \xrightarrow{D} \chi.$$

MULTIDIMENSIONAL BROWNIAN MOTION:

RECALL THAT GIVEN TWO RANDOM VARIABLES X AND Y, THE COVARIANCE OF X AND Y IS GIVEN BY

$$\begin{aligned}\operatorname{cov}(X,Y) &= E[(X-E[X])(Y-E[Y])] \\ &= E[XY - YE[X] - XE[Y] + E[X]E[Y]] \\ &= E[XY] - E[X]E[Y].\end{aligned}$$

LEMMA: IF X AND Y ARE INDEPENDENT, THEN

$$\operatorname{cov}(X,Y) = 0.$$

UNFORTUNATELY, THE CONVERSE IS NOT NECESSARILY TRUE:

SUPPOSE $X = \text{UNIFORM}(-1,1)$ AND LET $Y = X^2$

$$f_X(x) = \frac{1}{2} \quad \text{ON} \quad [-1,1].$$

$$F_Y(y) = P\{Y \le y\} = P\{X^2 \le y\} = P\{\pm X \le \sqrt{y}\}$$
$$= \frac{2\sqrt{y}}{2} = \sqrt{y} \quad \text{IF} \quad 0 \le y \le 1$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{2} y^{-1/2}$$

NOW, CLEARLY $E[X]=0$, AND THUS

$$\operatorname{cov}(XY) = E[XY] = E[X^3] = 0$$

HOWEVER, IF X AND Y ARE NORMAL RANDOM VARIABLES, AND

$\operatorname{cov}(X,Y)=0$ THEN X AND Y ARE INDEPENDENT.

IN FACT IF $X_1, X_2, \dots, X_n$ ARE NORMAL RANDOM VARIABLES AND

$\operatorname{cov}(X_j, X_k) = 0$ FOR EVERY $j \ne k$,

THEN $X_1, X_2, \dots, X_n$ ARE MUTUALLY INDEPENDENT.

DEF: $\{(B^1(t), B^2(t), \ldots, B^n(t)) : t \in [0,\infty)\}$ IS A STANDARD n-DIMENSIONAL BROWNIAN MOTION IF EACH OF ITS COMPONENTS ARE INDEPENDENT ONE-DIMENSIONAL BROWNIAN MOTIONS WITH $\sigma^2 = 1$.

SCALING RELATION:

IF c IS A POSITIVE REAL NUMBER, THEN

$$\{B(ct) : t \geq 0\} = \{c^{1/2} B(t) : t \geq 0\}.$$

THIS FOLLOWS, SINCE ALL PROPERTIES DEFINING BROWNIAN MOTION ARE EASILY VERIFIED, AND IF

$$B(t) = N(0, \sigma^2 t)$$

THEN

$$\begin{aligned} P\{B(ct) \leq x\} &= P\{N(0, \sigma^2 c t) \leq x\} \\ &= P\{c^{1/2} N(0, \sigma^2 t) \leq x\} \\ &= P\{c^{1/2} B(t) \leq x\} \quad \text{AS DESIRED.} \end{aligned}$$

COVARIANCE FORMULA: SUPPOSE $\{B(t) : t \geq 0\}$ IS A STANDARD B.M.

$$E[B_s B_t] = s \wedge t = \min\{s, t\}.$$

Pf: IF $s = t$, THEN $E[B_s^2] = \mathrm{VAR}(B_s) = s$, SO THE FORMULA WORKS IN THIS CASE.

NOW, IF $s < t$, THEN $B_t = B_s + (B_t - B_s)$

AND USING THE FACT THAT B_s AND $B_t - B_s$ ARE INDEPENDENT WITH MEAN 0, WE HAVE

$$\begin{aligned} E[B_s B_t] &= E[B_s^2] + E[B_s(B_t - B_s)] \\ &= s + 0 = s = s \wedge t \quad \checkmark \end{aligned}$$

HW: 6.6.9, 6.6.10, 6.6.11

§ 6.2 MARKOV PROPERTY, REFLECTION PRINCIPLE.

THM: (MARKOV PROPERTY): IF $s > 0$, AND $\{B(t) : t \geq 0\}$ IS A BROWNIAN MOTION, THEN $\{B(s+t) - B(s) : t \geq 0\}$ IS A BROWNIAN MOTION WHICH IS INDEPENDENT OF $B(r)$ IF $r \leq s$.

MORE PRECISELY, WE MAY SAY IF $0 \le s < t$, AND

$$0 \le r_1 < r_2 < \dots < r_n \le s$$

THEN $B(s+t) - B(s)$ IS INDEPENDENT OF THE VECTOR

$$(B(r_1), B(r_2), \dots, B(r_n)).$$

EVEN BETTER:

THM: (STRONG MARKOV PROPERTY). IF T IS A STOPPING TIME FOR $B(t)$, THEN

$\{B(T-t) - B(T) : t \ge 0\}$ IS A BROWNIAN MOTION

INDEPENDENT OF $B(r)$, $r \le T$.

WE HAVE NOT (AND WILL NOT) GIVE A PRECISE DEFINITION OF A STOPPING TIME FOR BROWNIAN MOTION, HOWEVER AS A CONCRETE EXAMPLE,

$$T_a = \min\{t \ge 0 : B(t) = a\}$$

IS A STOPPING TIME.

THM: $\{T_a : a \ge 0\}$ HAS STATIONARY, INDEPENDENT INCREMENTS. THAT IS,

i) IF $a < b$, THEN $T_b - T_a$ IS THE SAME AS T_{b-a}

ii) IF $a_0 = 0 < a_1 < \dots < a_n$, THEN

$T_{a_1} - T_{a_0}, \dots, T_{a_n} - T_{a_{n-1}}$ ARE INDEPENDENT.

Pf: IF $T = T_a$, THEN BY THE STRONG MARKOV PROPERTY,

$B(T+t) - B(T)$ IS A BROWNIAN MOTION INDEPENDENT OF B_r, IF $r \le T$

SINCE $B_T = a$, WE SEE $T_b - T_a$ IS JUST THE HITTING TIME OF $b-a$ FOR $B(T+t) - B(T)$, AND HENCE

$$T_b - T_a = T_{b-a} \quad (\text{IN DISTRIBUTION})$$

⋮

▽

TO PROVE ii) WE NOW SET $T = T_{a_{n-1}}$ TO SEE THAT $\{B(T+t) - B(T) : t \geq 0\}$ IS A BROWNIAN MOTION INDEPENDENT OF $B(r)$, $r \leq T$. SINCE $T_{a_n} - T_{a_{n-1}}$ IS DETERMINED BY

$$B(T+t) - B(t) \text{ FOR } t \geq 0,$$

WHILE $(T_{a_1} - T_{a_0}, \ldots, T_{a_{n-1}} - T_{a_{n-2}})$ IS DETERMINED BY

$B(r)$, $r \leq T$, WE MUST HAVE $T_{a_n} - T_{a_{n-1}}$

INDEPENDENT OF $(T_{a_1} - T_{a_0}, \ldots, T_{a_{n-1}} - T_{a_{n-2}})$.

INDUCTION COMPLETES THE PROOF. ✓

DISTRIBUTION OF T_a:

SUPPOSE $a > 0$.

$$P\{T_a \leq t\} = P\{T_a \leq t, B(t) > a\} + P\{T_a \leq t, B(t) < a\}$$

NOW, SINCE ANY PATH THAT STARTS AT 0 AND ENDS ABOVE $a > 0$ MUST CROSS a, SO

$$P\{T_a \leq t, B(t) > a\} = P\{B(t) > a\}$$

NEXT, WE NOTE THAT

$$B(T_a + t) - B(T_a) = B(T_a + t) - a$$

IS A BROWNIAN MOTION INDEPENDENT OF B_r, $r \leq T_a$, AND SINCE THE NORMAL DISTRIBUTION IS SYMMETRIC ABOUT 0, WE MUST HAVE

$$P\{T_a \leq t, B(t) < a\} = P\{T_a \leq t, B(t) > a\}$$

AND THUS

$$P\{T_a \leq t\} = 2P\{B(t) > a\}$$

$$= 2\int_a^\infty (2\pi t)^{-1/2} \exp(-x^2/2t)\, dt$$

NEXT, SETTING $x = \frac{t^{1/2} a}{s^{1/2}}$ $\quad dx = \frac{-1}{2} \frac{t^{1/2} a}{s^{3/2}} ds$

$$P\{T_a \le t\} = \int_0^t (2\pi t)^{-1/2} \exp\left\{\frac{-a^2}{2s}\right\} \frac{t^{1/2} a}{s^{3/2}} ds$$

$$= \int_0^t (2\pi s^3)^{-1/2} a \exp\left\{\frac{-a^2}{2s}\right\} ds$$

THUS, THE DENSITY OF T_a IS

$$(2\pi s^3)^{-1/2} a \exp\left\{\frac{-a^2}{2s}\right\}$$

<u>THM</u>: THE PROBABILITY THAT $B(t)$ HAS NO ZERO IN THE INTERVAL (s,t) IS GIVEN BY

$$\frac{2}{\pi} \arcsin \sqrt{\frac{s}{t}}$$

<u>Pf</u>: IF $B(s) = a > 0$, THEN THE PROBABILITY OF AT LEAST ONE ZERO IN (s,t) IS

$$P\{T_a \le t-s\}$$

SINCE $B(s) = N(0,s)$ WE HAVE

$$P\{T_a \le t-s\} =$$

$$P\{T_a \le t-s,\ B(s) > 0\} + P\{T_a \le t-s,\ B(s) < 0\}$$

$$= 2P\{T_a \le t-s,\ B(s) > 0\}$$

$$= 2\int_0^\infty \frac{1}{\sqrt{2\pi s}} e^{-a^2/2s} \int_0^{t-s} (2\pi r^3)^{-1/2} a \exp\left\{\frac{-a^2}{2r}\right\} dr\, da$$

$$= \frac{1}{\pi\sqrt{s}} \int_0^{t-s} r^{-3/2} \underbrace{\int_0^\infty a \exp\left\{-\frac{a^2}{2}\left[\frac{1}{s} + \frac{1}{r}\right]\right\} da}_{u = \frac{a^2}{2} \quad du = a\, da} dr$$

$$= \frac{1}{\pi\sqrt{s}} \int_0^{t-s} r^{-3/2} \int_0^\infty \exp\left\{-u\left[\frac{1}{s} + \frac{1}{r}\right]\right\} du\, dr$$

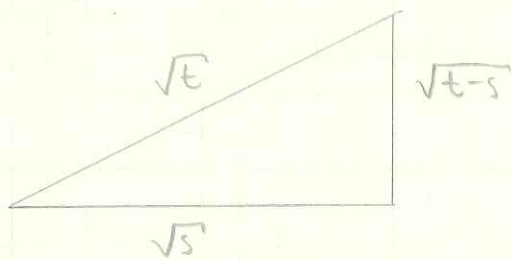
$$= \frac{1}{\pi\sqrt{s}} \int_0^{t-s} r^{-3/2} \left\{ \frac{1}{\frac{1}{s}+\frac{1}{r}} \right\} dr = \frac{1}{\pi\sqrt{s}} \int_0^{t-s} r^{-3/2} \left(\frac{sr}{s+r} \right) dr$$

$$= \frac{\sqrt{s}}{\pi} \int_0^{t-s} \frac{1}{\sqrt{r}\,(s+r)}\, dr = \frac{1}{\pi} \int_0^{\sqrt{(t-s)/s}} \frac{1}{sv(1+v^2)}\, 2sv\, dv$$

$$r = sv^2$$

$$dr = 2sv\,dv$$

$$= \frac{2}{\pi} \int_0^{\sqrt{(t-s)/s}} \frac{1}{1+v^2}\, dv = \frac{2}{\pi} \arctan\left(\sqrt{\frac{t-s}{s}} \right)$$



IF $\tan\theta = \dfrac{\sqrt{t-s}}{\sqrt{s}}$

THEN $\cos\theta = \dfrac{\sqrt{s}}{\sqrt{t}}$

THAT IS

$$P\{T_a \le t-s\} = \frac{2}{\pi} \arccos\left(\frac{\sqrt{s}}{\sqrt{t}} \right)$$

AND THUS THE PROBABILITY OF NO ZEROS IN (s,t) IS

$$1 - \frac{2}{\pi} \arccos\left(\frac{\sqrt{s}}{\sqrt{t}} \right) = \frac{2}{\pi} \left(\frac{\pi}{2} - \arccos\left(\frac{\sqrt{s}}{\sqrt{t}} \right) \right)$$

$$= \frac{2}{\pi} \arcsin\left(\frac{\sqrt{s}}{\sqrt{t}} \right). \checkmark$$

HW: 6.6.17, 6.6.18, 6.6.20

NEXT, LET US DEFINE

$L_t = \max\{ s < t : B(s) = 0 \}$ THE "LAST" ZERO BEFORE TIME t.

NOW, $P\{L_t < s\} = \dfrac{2}{\pi} \arcsin\sqrt{\dfrac{s}{t}}$, AND THUS

$$P\{L_t = s\} = \frac{d}{ds} P\{L_t < s\} = \frac{2}{\pi} \frac{1}{\sqrt{1-\frac{s}{t}}} \cdot \frac{1}{2\sqrt{st}} = \frac{1}{\pi} \frac{1}{\sqrt{s(t-s)}}.$$

NOTICE THAT

i) $P\{L_t = s\}$ IS SYMMETRIC ABOUT $t/2$

ii) $P\{L_t = s\} \longrightarrow \infty$ AS $s \longrightarrow 0$.

iii) $P\{L_t > 0\} = 1 - \frac{2}{\pi} \arcsin \sqrt{\frac{0}{t}} = 1$.

THIS LAST REMARK IS SURPRISING. LET

$$T_0^+ = \min\{s > 0 : B_s = 0\}$$

iii) IMPLIES FOR ALL $t > 0$ WE HAVE

$$P\{T_0^+ \leq t\} = P\{L_t > 0\} = 1$$

THUS, WE CAN FIND A SEQUENCE OF TIMES (s_n) WITH

$s_n < \frac{1}{n}$ AND $B(s_n) = 0$

THIS IMPLIES THAT BROWNIAN MOTION HAS INFINITELY MANY ZEROS!

§ 6.3: MARTINGALES, HITTING TIMES

THM: IF $\{B(t) : t \in [0, \infty)\}$ IS A BROWNIAN MOTION, THEN $B(t)$ IS A MARTINGALE. THAT IS,

$$E[B(t) \mid B(r) \text{ WITH } r \leq s] = B(s)$$

THAT IS, THE EXPECTATION OF $B(t)$ GIVEN THE ENTIRE HISTORY BEFORE s IS GIVEN BY $B(s)$. MORE SPECIFICALLY,

IF $0 \leq r_1 < \ldots < r_n < s$, THEN

$$E[B_t \mid B_s, B_{r_n}, \ldots, B_{r_1}] = B_s.$$

Pf: GIVEN $B(r)$, $r \leq s$, THE VALUE OF $B(s)$ IS KNOWN, WHILE $B(t) - B(s)$ IS INDEPENDENT OF r AND s, AND HAS MEAN 0. THUS

$$\begin{aligned} E[B_t \mid B_r, r \leq s] &= E[B_s + B_t - B_s \mid B_r, r \leq s] \\ &= E[B_s \mid B_r, r \leq s] + E[B_t - B_s \mid B_r \leq s] \\ &= B_s. \checkmark \end{aligned}$$

THM: (STOPPING THEOREM FOR BOUNDED MARTINGALES):

SUPPOSE $\{B(t) : t \in [0,\infty)\}$ IS A BROWNIAN MOTION, AND T IS A STOPPING TIME WITH

$$P\{T < \infty\} = 1$$

AND $|M_{T\wedge t}| \le K$ FOR ALL t. THEN,

$$E[M_T] = E[M_0].$$

DEFINE THE EXIT TIME FOR THE INTERVAL (a,b) BY

$$\tau = \min\{t : B(t) \notin (a,b)\} \qquad a < 0 < b$$

THEN,

$$P\{B_\tau = b\} = \frac{-a}{b-a} \quad \text{AND} \quad P\{B_\tau = a\} = \frac{b}{b-a}$$

RECALL THAT THE DENSITY OF T_b IS

$$P\{T_b = s\} = (2\pi s^3)^{-1/2}\, b \exp\left\{\frac{-b^2}{2s}\right\}$$

AND $P\{T_b < \infty\} = \int_0^\infty (2\pi s^3)^{-1/2}\, b \exp\left\{\frac{-b^2}{2s}\right\} ds$

$$t = s^{-1/2}$$

$$dt = -\frac{1}{2} s^{-3/2}\, ds \qquad d$$

$$= \frac{2b}{\sqrt{2\pi}} \int_0^\infty \exp\{-b^2 t\}\, dt = 1.$$

CLEARLY $\tau < T_b$, AND THUS $P\{\tau < \infty\} = 1$.

NOW, $|B(\tau \wedge t)| \le \max\{|a|, b\}$ AND THUS WE MAY USE THE THEOREM. IT FOLLOWS THAT

$$0 = E[B(0)] = a P\{B_\tau = a\} + b P\{B_\tau = b\}$$

THUS,

$$0 = a\,P\{B_\tau = a\} + b\left[1 - P\{B_\tau = a\}\right]$$

$$\Rightarrow P\{B_\tau = a\} = \frac{b}{b-a}$$

AND SIMILARLY $P\{B_\tau = b\} = \dfrac{-a}{b-a}$.

WE ALSO PROVED IN PASSING THE FOLLOWING THEOREM:

<u>THM</u>: FOR ALL a, $P\{T_a < \infty\} = 1$

A QUADRATIC MARTINGALE:

<u>THM</u>: $B(t)^2 - t$ IS A MARTINGALE. THAT IS, IF $s < t$, THEN

$$E\left[B_t^2 - t \mid B_r, r \le s\right] = B_s^2 - s.$$

<u>Pf</u>:

$$E\left[B_t^2 \mid B_r, r \le s\right] = E\left[\{B_s + B_t - B_s\}^2 \mid B_r, r \le s\right]$$

$$= E\left[B_s^2 \mid B_r, r \le s\right] + 2E\left[B_s(B_t - B_s) \mid B_r, r \le s\right] + E\left[(B_t - B_s)^2 \mid B_r, r \le s\right]$$

$$= B_s^2 + 2B_s E\left[B_t - B_s \mid B_r, r \le s\right] + E\left[(B_t - B_s)^2 \mid B_r, r \le s\right]$$

$$= B_s^2 + 2B_s E\left[B_t - B_s\right] + E\left[(B_t - B_s)^2\right]$$

$$= B_s^2 + 0 + t - s \checkmark$$

MEAN EXIT TIME FROM AN INTERVAL: GIVEN $a < 0 < b$, LET

$$\tau = \min\{t : B_t \in (a,b)\}.$$

THEN $E[\tau] = -ab$.

NOW, $B_t^2 - t$ IS NOT BOUNDED, BUT APPLYING THE THEOREM ANYWAYS,

$$0 = E[B_\tau^2 - \tau] = b^2\left(\frac{-a}{b-a}\right) + a^2\left(\frac{b}{b-a}\right) - E[\tau]$$

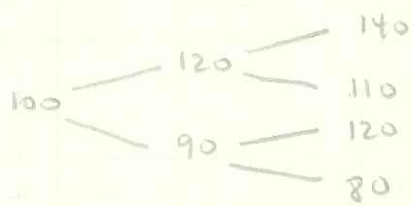
THUS,

$$E[\tau] = \frac{a^2 b - ab^2}{b-a} = -ab. \checkmark$$

§ 6.4 OPTION PRICING IN DISCRETE TIME

SUPPOSE WE ARE MONITORING A STOCK, AND AT THE END OF EACH DAY THE STOCK WILL BE EITHER UP 20 POINTS, OR DOWN 10 POINTS.

IF THE STOCK STARTS AT 100, THE POSSIBLE OUTCOMES AT THE END OF DAY TWO ARE GIVEN AS FOLLOWS:



A <u>EUROPEAN CALL OPTION</u> WITH STRIKE PRICE 100 AND EXPIRY 2 MEANS WE HAVE THE OPTION OF BUYING THE STOCK FOR 100 AT TIME 2, THAT IS AFTER SEEING THE OUTCOME AFTER THE FIRST TWO STAGES.

IN THIS SITUATION, WE WILL NOT BUY IF THE STOCK IS 80, BUT IN THE OTHER CASES WE WILL MAKE THE PURCHASE AND IMMEDIATELY SELL IT TO MAKE 10, 20 OR 40 IN PROFIT.

THAT IS, OUR PROFIT IS

$$(X_2 - 100)^+ = \max\{X_2 - 100, 0\}$$

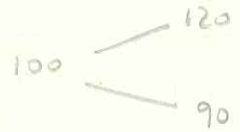
WHERE X_2 IS THE PRICE AT TIME 2.

WHAT SHOULD WE CHARGE FOR THIS OPTION?

THIS WOULD SEEM TO BE IMPOSSIBLE WITHOUT KNOWLEDGE OF THE PROBABILITIES OF THE OUTCOMES?

HOWEVER THERE <u>IS</u> A METHOD TO PRICE THE OPTION.

LOOKING AT PART OF THE TREE,



SUPPOSE WE PAY C FOR THE OPTION, THEN WE EITHER MAKE A PROFIT OF $20 - C$, OR WE ARE OUT C EUROS. THAT IS,

	STOCK	OPTION PROFIT
UP	20	$20 - C$
DOWN	-10	$-C$

NOW SUPPOSE WE BUY x UNITS OF STOCK AND y UNITS OF THE OPTION (WITH NEGATIVE UNITS MEANING WE SOLD INSTEAD OF BOUGHT). ONE POSSIBLE STRATEGY IS TO CHOOSE x AND y SO THAT THE OUTCOME IS THE SAME IF THE STOCK GOES UP OR DOWN.

$$20x + (20 - c)y = -10x + (-c)y$$

$$\Rightarrow \quad 30x + 20y = 0 \quad \Rightarrow \quad y = -\frac{3}{2}x$$

PLUGGING THIS INTO $-10x - cy$ YIELDS

$$-10x + \frac{3}{2}cx = \left(-10 + \frac{3}{2}c\right)x$$

THUS IF $\frac{3}{2}c > 10$, THAT IS $c > \frac{20}{3}$

THEN WE CAN MAKE A PROFIT WITH NO RISK BY BUYING LARGE AMOUNTS OF STOCK AND SELLING 1.5 TIMES AS MANY OPTIONS FOR A PRICE OF AT LEAST $20/3$.

NOTICE NO RISK IS INVOLVED!

<u>DEF</u>: A SCHEME TO MAKE MONEY WITHOUT RISK IS CALLED AN <u>ARBITAGE</u> OPPORTUNITY.

Q: DO ARBITAGE OPPORTUNITIES REALLY EXIST?

GENERALIZING SUPPOSE WE HAVE m STOCKS AND

$a_{i,j}$ IS THE PROFIT FOR THE i^{TH} STOCK WHEN THE j^{TH} OUTCOME OCCURS.

THM: EXACTLY ONE OF THE FOLLOWING OCCURS

i) THERE IS A BUYING SCHEME $\vec{x} = (x_1, x_2, \ldots, x_m)$ SO THAT

$$\sum_{i=1}^{m} x_i a_{i,j} \geq 0 \quad \text{FOR EACH } j \text{ AND}$$

$$\sum_{i=1}^{m} x_i a_{i,k} > 0 \quad \text{FOR SOME } k$$

ii) THERE IS A PROBABILITY VECTOR $p_j > 0$ SO THAT

$$\sum_{j=1}^{n} a_{i,j} p_j = 0 \quad \text{FOR ALL } i.$$

NOTICE THAT i) CORRESPONDS TO AN ARBITRAGE OPPORTUNITY SINCE WE NEVER LOSE MONEY, BUT IN AT LEAST ONE OUTCOME WE MAKE MONEY.

THE VECTOR $\vec{p} = (p_1, p_2, \ldots, p_n)$ IS CALLED A MARTINGALE VECTOR (OR MEASURE) SINCE THE EXPECTED CHANGE IN THE i^{TH} STOCK IS 0 FOR ALL i).

Pf: IF i) IS TRUE, THEN FOR ANY STRICTLY POSITIVE PROBABILITY VECTOR,

$$\sum_{j=1}^{n} p_j \sum_{i=1}^{m} x_i a_{i,j} > 0$$

$$= \sum_{i=1}^{m} x_i \sum_{j=1}^{n} a_{i,j} p_j > 0$$

$$\Rightarrow \sum_{j=1}^{n} a_{i,j} p_j \neq 0 \quad \text{FOR SOME } i.$$

NOW SUPPOSE i) IS FALSE

THE SET OF LINEAR COMBINATIONS

$$\sum_{i=1}^{m} x_i a_{ij}$$ FORM A LINEAR SUBSPACE OF THE n-DIMENSIONAL VECTOR SPACE SPANNED BY $x_1, \ldots, x_m$

CALL THIS SUBSPACE $\mathcal{L}$, AND LET

$$\mathcal{O} = \{ \vec{y} : y_j \geq 0 \text{ FOR ALL } j \}$$

IF i) IS FALSE THEN $\mathcal{L}$ INTERSECTS $\mathcal{O}$ ONLY AT THE ORIGIN

WE MAY EXTEND $\mathcal{L}$ TO AN $(n-1)$-DIMENSIONAL SUBSPACE $\mathcal{H}$ WHICH INTERSECTS $\mathcal{O}$ ONLY AT THE ORIGIN, IN WHICH CASE

$$\mathcal{H} = \{ \vec{y} : \sum_{j=1}^{n} y_j p_j = 0 \}$$

FOR SOME $p_1, p_2, \ldots, p_j$. SINCE $\mathcal{L} \subseteq \mathcal{H}$, WE HAVE

$a_{i,j} \in \mathcal{H}$ FOR EACH FIXED i, $j = 1, \ldots, n$

THUS

$$\sum_{j=1}^{n} a_{ij} p_j = 0$$

IT REMAINS TO SHOW THAT THE $p_j > 0$. BUT IF $p_j < 0$ FOR SOME j, LET i BE SUCH THAT $p_i > 0$

THEN

$$p_j - \left(\frac{p_j}{p_i} \right) p_i = 0$$

AND THUS THE VECTOR

$$\vec{y} = (0, \ldots, 1, \ldots, -\frac{p_j}{p_i}, \ldots) \in \mathcal{H} \cap \mathcal{O}$$

j^{TH} SLOT $\qquad$ i^{TH} SLOT

WHICH IS A CONTRADICTION. ✓

		$j=1$	$j=2$
STOCK	$i=1$	30	-10
OPTION	$i=2$	$20-c$	$-c$

IF THERE IS NO ARBITRAGE, THEN THERE MUST BE AN ASSIGNMENT OF PROBABILITIES p_j SO THAT

$$30 p_1 - 10 p_2 = 0 \qquad (20-c) p_1 + (-c) p_2 = 0$$

FROM THE FIRST EQUATION WE GET

$$p_1 = 1/4 \qquad p_2 = 3/4 \qquad c = 5$$

NOTICE THAT THE FIRST EQUATION IS JUST THE FACT THAT, WITH THESE VALUES OF p_i THE CHANGE OF STOCK PRICE IS A MARTINGALE

THE SECOND EQUATION IS THEN THE EXPECTED VALUE UNDER THESE PROBABILITIES OF THE PRICE OF THE STOCK.

§ 6.5 THE BLACK-SCHOLES FORMULA

<u>THM</u>: SUPPOSE $\{B_t : t \in [0, \infty)\}$ IS A BROWNIAN MOTION, AND θ IS A REAL NUMBER. THEN

$$\exp\left\{\theta B_t - t\frac{\theta^2}{2}\right\} \text{ IS A MARTINGALE.}$$

THAT IS,

$$E\left[\exp\left\{\theta B_t - t\frac{\theta^2}{2}\right\} \middle| B_r, r \le s\right] = \exp\left\{\theta B_s - s\frac{\theta^2}{2}\right\}.$$

FIRST WE NEED A COUPLE FACTS.

i) $$-\frac{x^2}{2u} + \theta x = -\frac{(x - u\theta)^2}{2u} + \frac{u\theta^2}{2}$$

ii) $$E[\exp(\theta B_u)] = \int_{\mathbb{R}} e^{\theta x} \frac{1}{\sqrt{2\pi u}} e^{-x^2/2u}\, dx$$

$$= \frac{e^{u\theta^2/2}}{\sqrt{2\pi u}} \int_{\mathbb{R}} \exp\left\{-\frac{(x-u\theta)^2}{2u}\right\} dx = e^{u\theta^2/2}$$

NOW TO PROVE THE THEOREM.

<u>Pf</u>: AS USUAL, WE WRITE $B_t = B_s + B_t - B_s$, AND THUS

$$E\left[\exp\left\{\theta B_t - t\frac{\theta^2}{2}\right\} \middle| B_r, r \le s\right]$$

$$= \exp\left\{-\frac{t\theta^2}{2}\right\} E\left[\exp\{\theta B_t\} \middle| B_r, r \le s\right]$$

$$= \exp\left\{-\frac{t\theta^2}{2}\right\} E\left[\exp\{\theta B_s\} \exp\{\theta(B_t - B_s)\} \mid B_r, r \le s\right]$$

$$= \exp\left\{-\frac{t\theta^2}{2}\right\} E\left[\exp\{\theta B_s\} \mid B_r, r \le s\right] \times E\left[\exp\{\theta(B_t - B_s)\} \mid B_r, r \le s\right]$$

$$= \exp\left\{-\frac{t\theta^2}{2} + \theta B_s\right\} E\left[\exp\{\theta B_{t-s}\}\right]$$

$$= \exp\left\{-\frac{t\theta^2}{2} + \theta B_s\right\} \exp\left\{(t-s)\frac{\theta^2}{2}\right\}$$

$$= \exp\left\{\theta B_s - \frac{s\theta^2}{2}\right\}$$ AS DESIRED. ✓

ONE MODEL FOR A STOCK PRICE IS

$$X_t = X_0 \exp\{\mu t + \sigma B_t\}$$

WHERE μ IS THE EXPONENTIAL GROWTH RATE OF THE STOCK AND σ IS ITS VOLATILITY

(HERE WE ASSUME μ AND σ ARE CONSTANT)

THE DISCOUNTED STOCK RATE IS

$$e^{-rt} X_t = X_0 \exp\{(\mu - r)t + \sigma B_t\}$$

WHICH AJUSTS THE PRICE OF THE STOCK AT TIME t TO TODAYS DOLLARS BY TAKING INTO ACCOUNT AN INFLATION RATE r.

WE SEE IMMEDIATELY THAT IF

$$\mu = r - \frac{\sigma^2}{2}, \quad \text{THAT IS} \quad \mu - r = -\frac{\sigma^2}{2}$$

THEN THE DISCOUNTED STOCK PRICE IS A MARTINGALE.

NOW CONSIDER A EUROPEAN CALL OPTION WITH STRIKE PRICE K AND EXPIRY T.

IF WE HAVE PURCHASED THE OPTION, THEN AT TIME T WE WILL HAVE

$$\max\{X_T - K, 0\} \text{ EUROS IN HAND.}$$

THM: THE (FAIR) PRICE OF A EUROPEAN CALL OPTION WITH STRIKE PRICE K AND EXPIRY T IS GIVEN BY

$$X_0\,\Phi(\sigma\sqrt{T} - \alpha) - e^{-rt} K\,\Phi(-\alpha)$$

WHERE
$$\Phi(t) = \int_{-\infty}^{t} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx,$$

$$\alpha = \frac{1}{\sigma\sqrt{T}} \log\left\{\frac{K}{X_0 e^{\mu t}}\right\}, \quad \mu = r - \frac{\sigma^2}{2}.$$

Pf: THE FAIR PRICE C IS GIVEN BY

$$C = E\left[e^{-rT}(X_T - K)^+\right].$$

NOTICE THAT

$$\log\left(\frac{X_T}{X_0}\right) = \log(X_T) - \log(X_0)$$

$$= \log\left\{X_0 \exp\{\mu T + \sigma B_T\}\right\} - \log X_0$$

$$= \mu T + \sigma B_T$$ WHICH IS NORMALLY DISTRIBUTED WITH MEAN μT AND VARIANCE $\sigma^2 T$.

THUS,

$$C = e^{-rT} \int_{\log(K/X_0)}^{\infty} (X_0 e^y - K)\frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-(y-\mu t)^2/2\sigma^2 t}\,dy$$

$$= e^{-rT} \int_{\log K - \log(X_0)}^{\infty} (X_0 e^y - K)\frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-(y-\mu t)^2/2\sigma^2 t}\,dy$$