

Pacific Northwest Number Theory Conference
Conjugate Reciprocal Polynomials with all
Roots on the Unit Circle

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Acknowledgments

Joint work with Kathleen Petersen (Queen's University).



Definitions

A polynomial $f \in \mathbb{C}[x]$ is *conjugate reciprocal* (CR) if

$$f(\overline{x}) = x^N \overline{f(1/\overline{x})}, \quad N = \deg f.$$

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- *reciprocal*: $c_{N-n} = c_n$
- *self-inversive*: $c_{N-n} = \xi \overline{c_n}$, for some $|\xi| = 1$.

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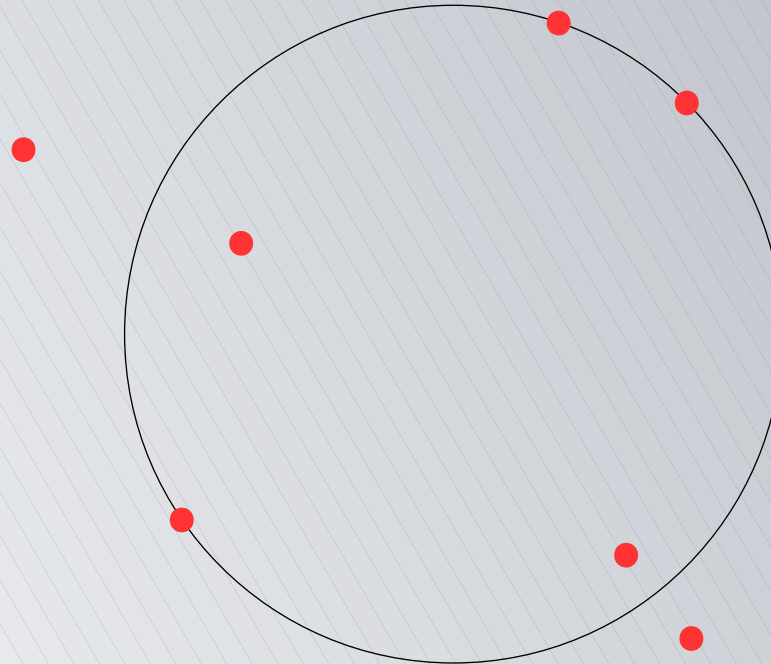
The CR condition implies

$$f(\alpha) = 0 \quad \Rightarrow \quad f(1/\bar{\alpha}) = 0$$

Moreover $\alpha = 1/\bar{\alpha}$ if and only if $\alpha \in \mathbb{T}$.

Thus the roots of f are either on $\mathbb{T}$ or come in pairs invariant under inversion across $\mathbb{T}$ (hence the term self-inversive).

Definitions



A plot of the roots of

$$x^7 + \left(\frac{1}{2} - \frac{i}{3}\right)x^6 - \left(\frac{1}{3} - \frac{i}{2}\right)x^5 + \frac{3}{2}x^4 + \frac{3}{2}x^3 - \left(\frac{1}{3} + \frac{i}{2}\right)x^2 + \left(\frac{1}{2} + \frac{i}{3}\right)x + 1$$

More Definitions

We will be primarily interested in monic CR polynomials. That is, polynomials of the form

$$f(x) = x^N + 1 + \sum_{n=1}^{N-1} c_n x^{N-n} \quad c_{N-n} = \overline{c_n}$$

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The number of real numbers needed to describe f is

$$N - 1 = \begin{cases} 2 \left(\frac{N-1}{2} \right) & N \text{ odd} \\ 2 \left(\frac{N-2}{2} \right) + 1 & N \text{ even.} \end{cases}$$

Yet More Definitions

We may identify monic CR polynomials of degree N with $\mathbb{R}^{N-1}$. To do this define the $N - 1 \times N - 1$ matrix X_N by

$$X_N[j, k] = \begin{cases} \frac{\sqrt{2}}{2} (\delta_{j,k} + \delta_{N-j,k}) & \text{if } 1 \leq j < \lfloor N/2 \rfloor \\ \delta_{j,k} & \text{if } j = N/2 \\ \frac{\sqrt{2}}{2} (i\delta_{N-j,k} - i\delta_{j,k}) & \text{if } \lfloor N/2 \rfloor < j < N \end{cases}$$

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$$X_5 = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ 1 & 0 & 0 & -i \end{bmatrix}$$

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Notice if $\mathbf{a} \in \mathbb{R}^{N-1}$ then $X_N \mathbf{a} =: \mathbf{c} \in \mathbb{C}^{N-1}$ and $c_{N-n} = \overline{c_n}$.

Yet More Definitions

Since $\mathbf{c} = X_N \mathbf{a}$ has the property

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$$\mathbf{a}(x) := x^N + 1 + \sum_{n=1}^{N-1} c_n x^{N-n}.$$

We explicitly identify the set of monic CR polynomials of degree N with $\mathbb{R}^{N-1}$ by the map

$$\mathbf{a} \mapsto \mathbf{a}(x).$$

The Definition of W_N

Finally we define,

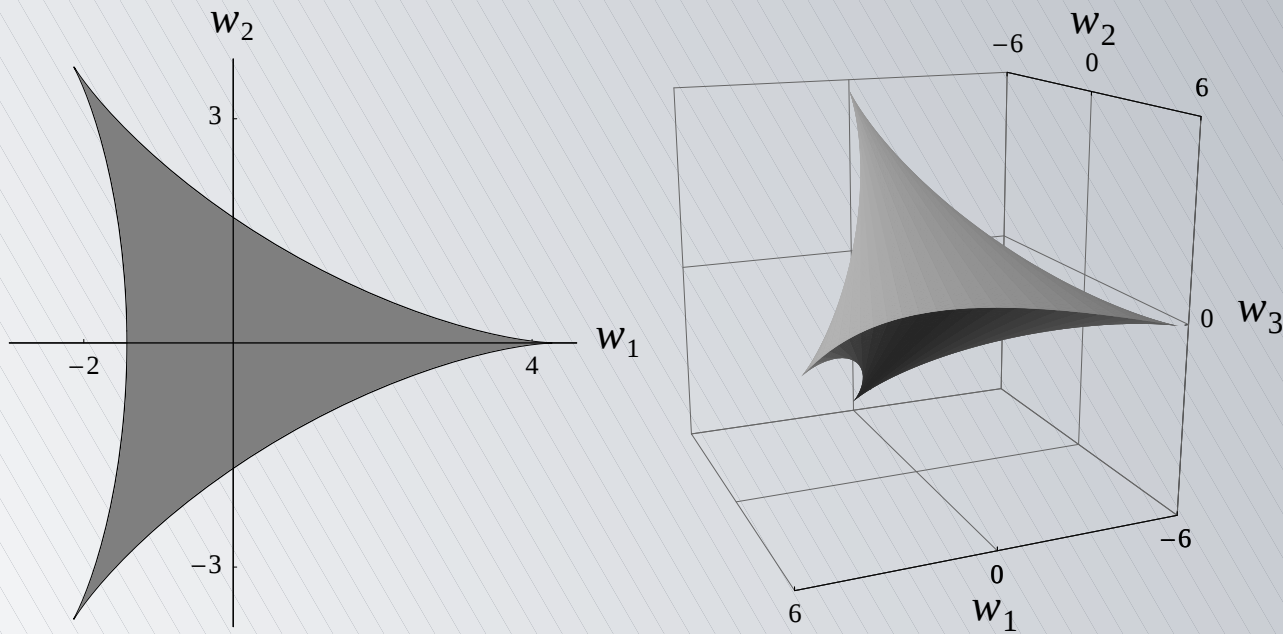
$$W_N = \{\mathbf{a} \in \mathbb{R}^{N-1} : \mathbf{a}(x) \text{ has all roots on } \mathbb{T}\}.$$

That is, W_N is the collection of vectors in $\mathbb{R}^{N-1}$ which correspond to degree N CR polynomials with all roots on the unit circle.

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Plots of W_3 and W_4 .

Theorems

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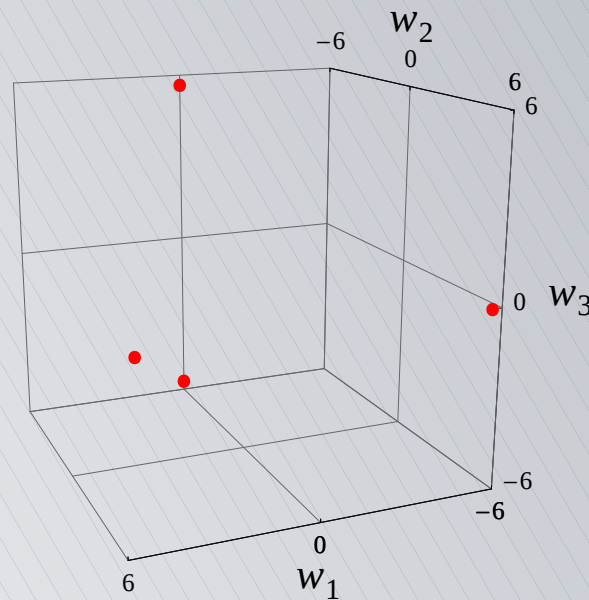
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Points in W_4 corresponding to the partition $\{4\}$.



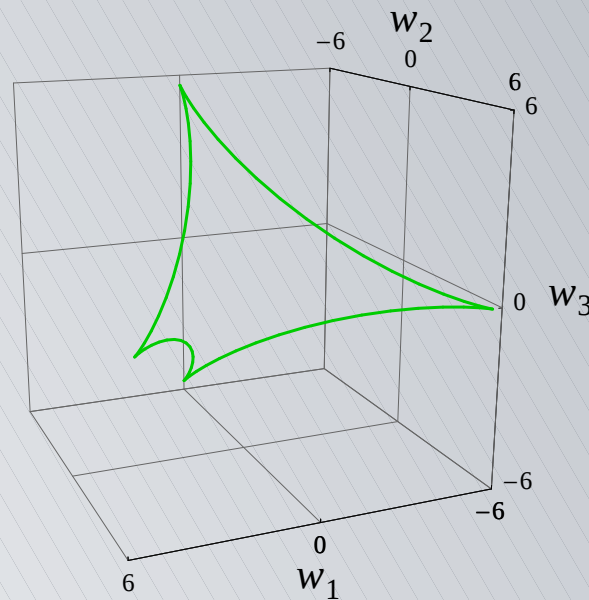
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Points in W_4 corresponding to the partition $\{3, 1\}$.



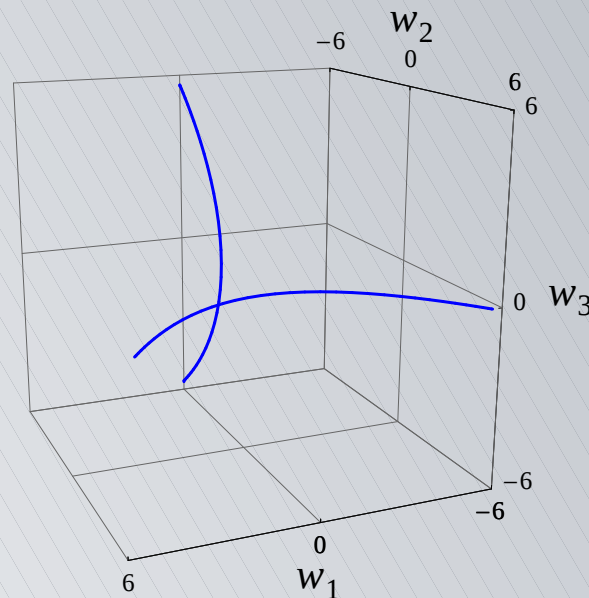
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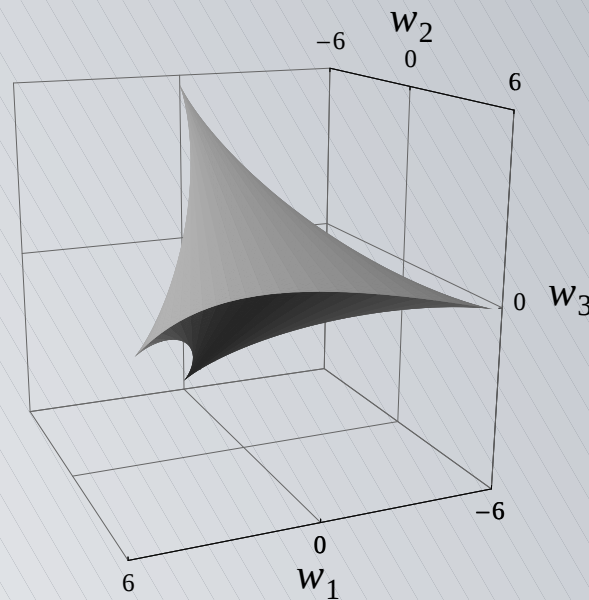
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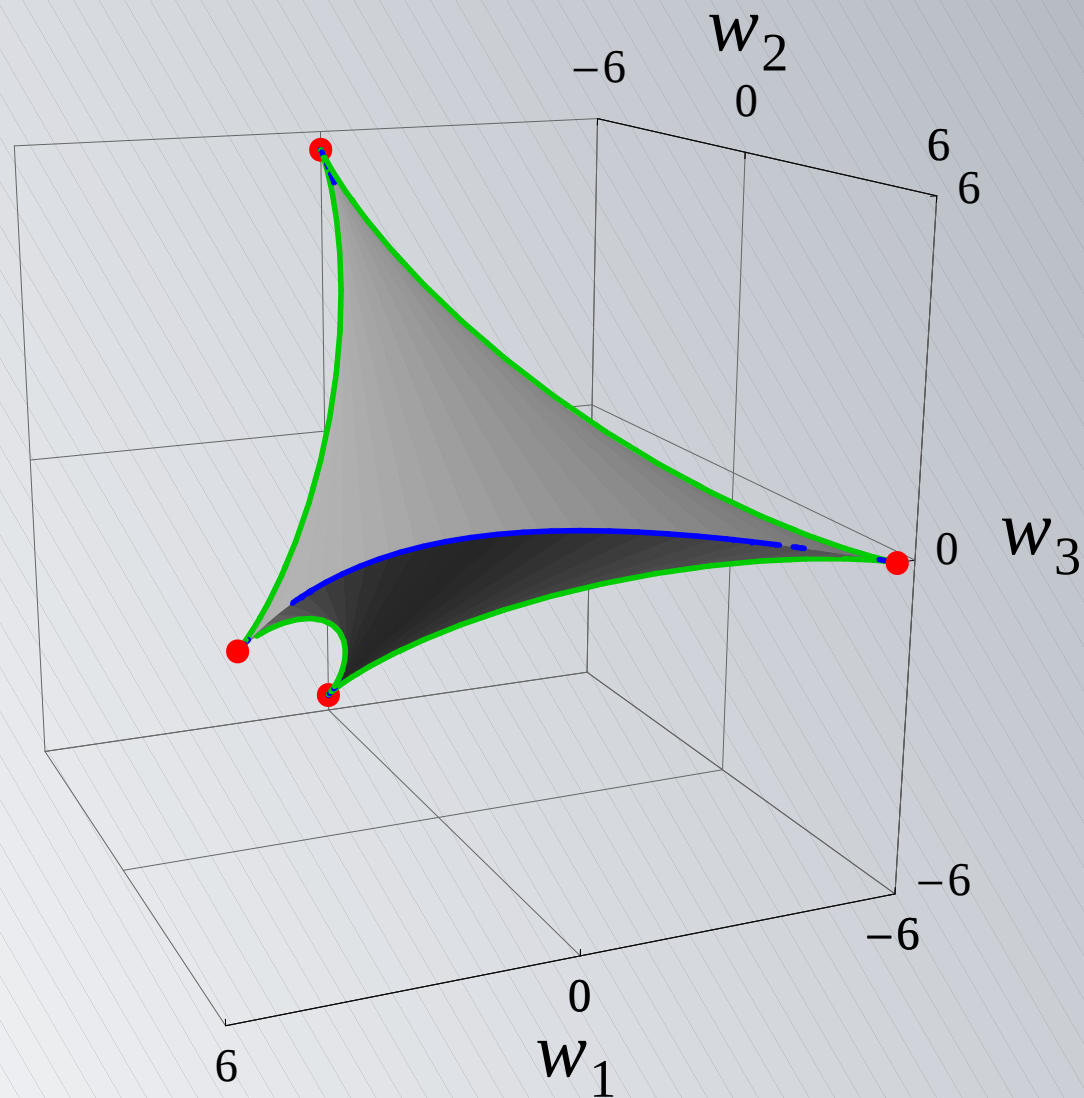
Theorem 2. W_N has the structure of a coloured $N - 1$ simplex where the colouring of $\mathbf{a} \in W_N$ is $\mathcal{P}(\mathbf{w})$.

Points in W_4 corresponding to the partition $\{1, 1, 1, 1\}$.



W_4 Coloured

$\{4\},$
 $\{3, 1\},$
 $\{2, 2\},$
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Theorem 3. *The group of isometries of W_N is isomorphic to D_N , the dihedral group of order $2N$.*

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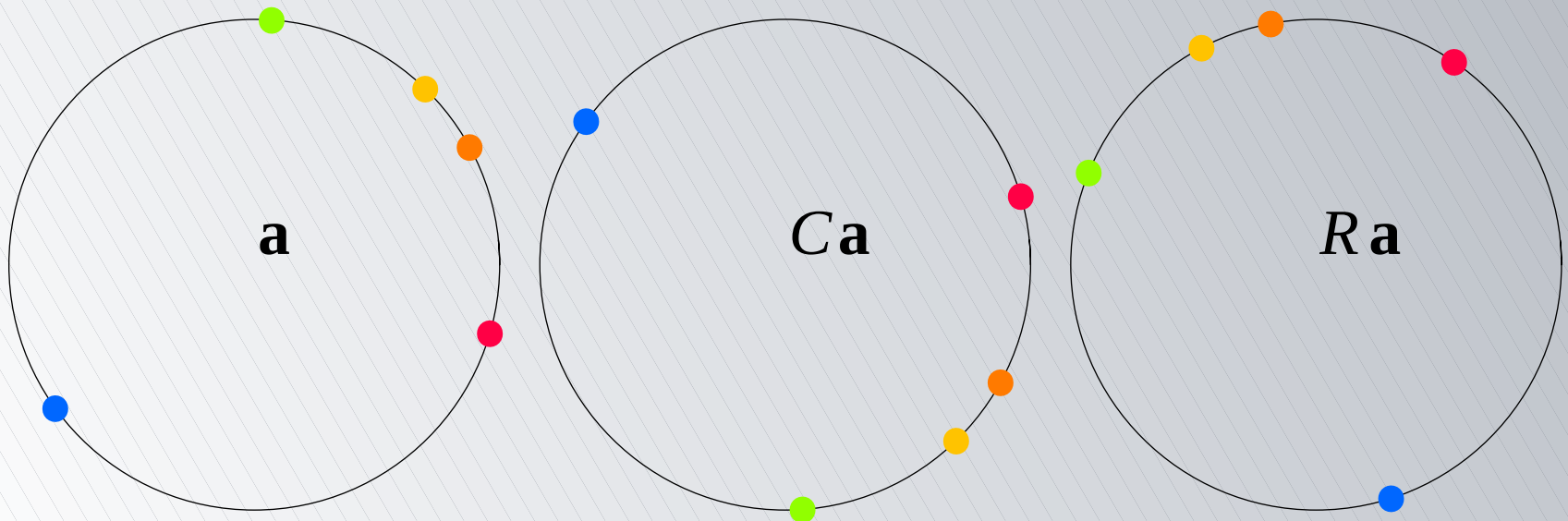
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That is, the volume of W_N is equal to the volume of the $N - 1$ dimensional ball of radius 2.

A Few Preliminaries

Proposition 5. *The vector $\mathbf{w}$ is in W_N if and only if*

$$\mathbf{w}(x) = \prod_{n=1}^N (x - \xi_n),$$

where $\xi_1, \xi_2, \dots, \xi_N$ are elements of $\mathbb{T}$ satisfying $\xi_1 \xi_2 \cdots \xi_N = (-1)^N$.

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Proof.

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It follows that $\mathbf{w}$ is CR and in W_N . The converse is obvious since every element of W_N is a polynomial with all roots on the unit circle and constant coefficient 1.

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Proposition 6. *The points in W_N corresponding to the partition $\{N\}$ are given by*

$$\mathbf{v}_n(x) = (x + \zeta_N^n)^N \quad n = 1, 2, \dots, N,$$

where $\zeta_N = e^{2\pi i/N}$.

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Proposition 7. *The points in W_N corresponding to the partition $\{N\}$ are given by*

$$\mathbf{v}_n(x) = (x + \zeta_N^n)^N \quad n = 1, 2, \dots, N,$$

where $\zeta_N = e^{2\pi i/N}$. Moreover, if $\mathbf{w} \in W_N$ then

$$\|\mathbf{w}\|^2 \leq \binom{2N}{N} - 2,$$

with equality if and only if $\mathbf{w} = \mathbf{v}_n$ for some $1 \leq n \leq N$.

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Corollary 7. *Each Isometry of W_N must map the set of vertices to itself. That is, $\text{Isom}(W_N)$ is a subgroup of S_N .*

Isometries and Parseval's Formula

To show R is an isometry of W_N we need to show for any points $\mathbf{w}_1, \mathbf{w}_2 \in W_N$,

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Now, since $R\mathbf{w}(x) = \mathbf{w}(\zeta_N^{-1}x)$,

$$\|R\mathbf{w}_1 - R\mathbf{w}_2\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |\mathbf{w}_1(\zeta_N^{-1}e^{i\theta}) - \mathbf{w}_2(\zeta_N^{-1}e^{i\theta})|^2 d\theta.$$

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The Distance Between Vertices

Lemma 8. *For each $1 \leq M \leq N$, the vertices farthest from $\mathbf{v}_M$ are $\mathbf{v}_{M+1}$ and $\mathbf{v}_{M-1}$.*

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This lemma implies that $\text{Isom}(W_N) \equiv D_N$.

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- T is in the group spanned by R and C .

Self Inversive Polynomials

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As in the CR case, the set of monic ω -CR polynomials is in bijective correspondence with $\mathbb{R}^{N-1}$.

There exists a linear map $X_{N,\omega} : \mathbb{R}^{N-1} \rightarrow \mathbb{C}^{N-1}$ so that

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We define

$$W_{N,\omega} = \left\{ \mathbf{a} \in \mathbb{R}^{N-1} : (x^N + \omega) + \sum_{n=1}^{N-1} c_n x^{N-n} \text{ has all roots on } \mathbb{T} \right\}$$

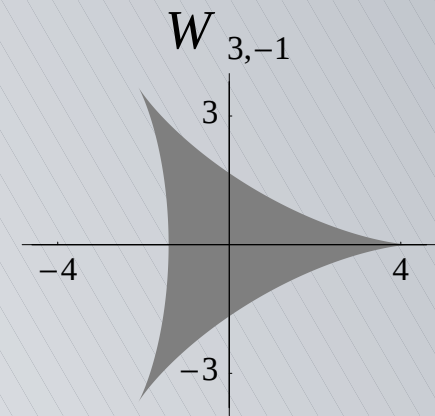
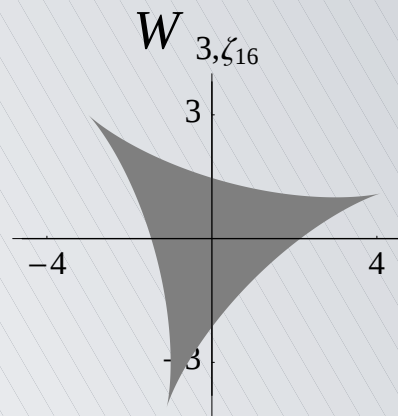
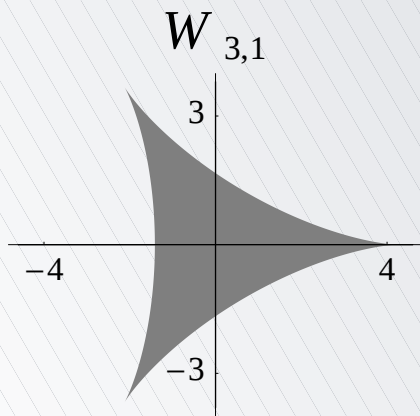
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A Change of Variables

Proposition 8. *The vector $\mathbf{w}$ is in $W_{N,\omega}$ if and only if*

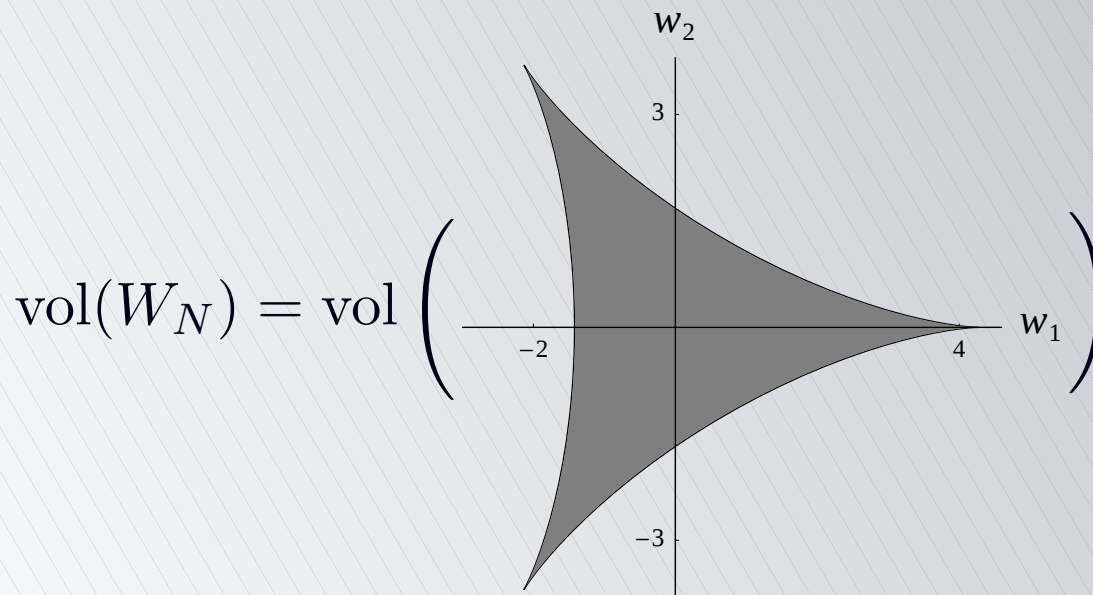
$$\mathbf{w}(x) = \prod_{n=1}^N (x - \xi_n),$$

where $\xi_1, \xi_2, \dots, \xi_N$ are elements of $\mathbb{T}$ satisfying $\xi_1 \xi_2 \cdots \xi_N = (-1)^N \omega$.

We define the map $E_{N,\omega} : \mathbb{T}^{N-1} \rightarrow W_{N,\omega}$ specified by

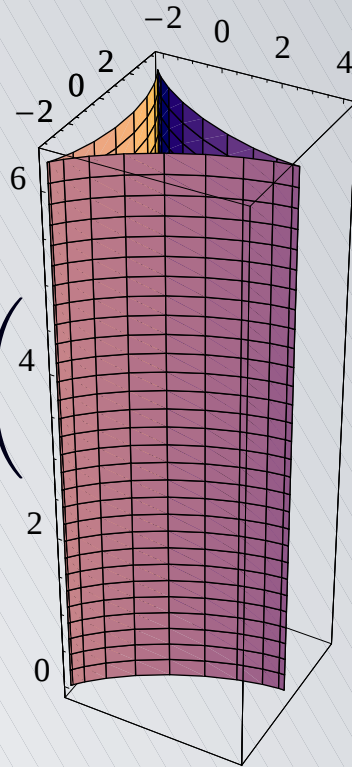
$\mathbf{a} = E_{N,\omega}(\boldsymbol{\xi}) = X_{N,\omega}^{-1} \mathbf{c}$ where $\mathbf{c}$ is obtained from $\boldsymbol{\xi}$ by

$$\left(x - \frac{(-1)^N \omega}{\xi_1 \xi_2 \cdots \xi_{N-1}} \right) \prod_{n=1}^{N-1} (x - \xi_n) = (x^N + \omega) + \sum_{n=1}^{N-1} c_n x^{N-n}.$$



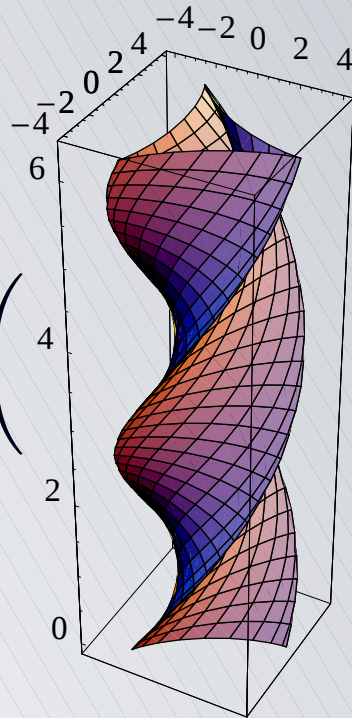
The Volume of W_N

$$\text{vol}(W_N) = \frac{1}{2\pi} \text{vol} \left(\begin{array}{c} 4 \\ 2 \\ 0 \end{array} \right)$$



The Volume of W_N

$$\text{vol}(W_N) = \frac{1}{2\pi} \text{vol} \left(\begin{array}{c} \text{3D plot of a twisted surface} \end{array} \right)$$



The Volume of W_N

$$\text{vol}(W_N) = \frac{1}{2\pi} \text{vol} \left(\text{cruller} \right)$$

cruller.mov

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That is,

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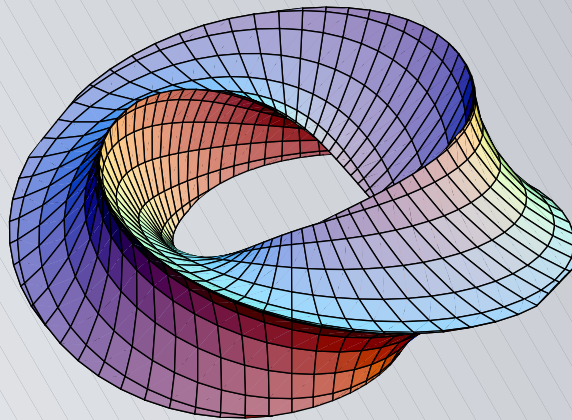
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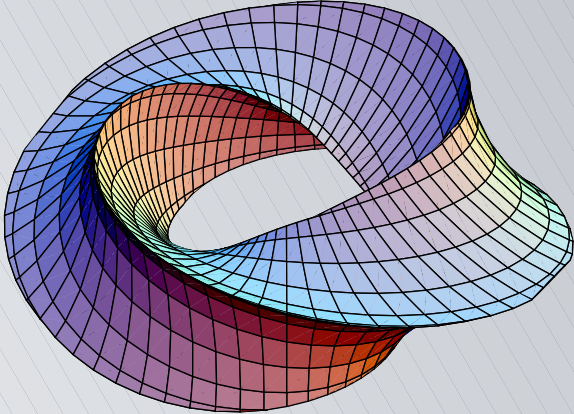
$$\text{vol}(W_N) = \frac{1}{2\pi} \text{vol} \left(\text{Figure} \right)$$



That is,

$$\text{vol}(W_N) = \frac{1}{2\pi N!} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{1 \leq m < n \leq N} |e^{i\theta_n} - e^{i\theta_m}| d\theta_1 \cdots d\theta_N.$$

The Volume of W_N

$$\text{vol}(W_N) = \frac{1}{2\pi} \text{vol} \left(\text{Figure} \right)$$


Dyson computed this integral in the context of RMT, and

$$\text{vol}(W_N) = \frac{2^{N-1} \pi^{(N-1)/2}}{\Gamma(\frac{N+1}{2})}.$$