Number Theory and Polynomials

Conjugate Reciprocal Polynomials with all Roots on the Unit Circle

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Definitions

A polynomial $f \in \mathbb{C}[x]$ is *conjugate reciprocal* (CR) if

$$f(\overline{x}) = x^N \overline{f(1/\overline{x})}, \qquad N = \deg f.$$

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- *reciprocal:* $c_{N-n} = c_n$
- *self-inversive:* $c_{N-n} = \xi\,\overline{c_n}$, for some $|\xi| = 1$.

Definitions

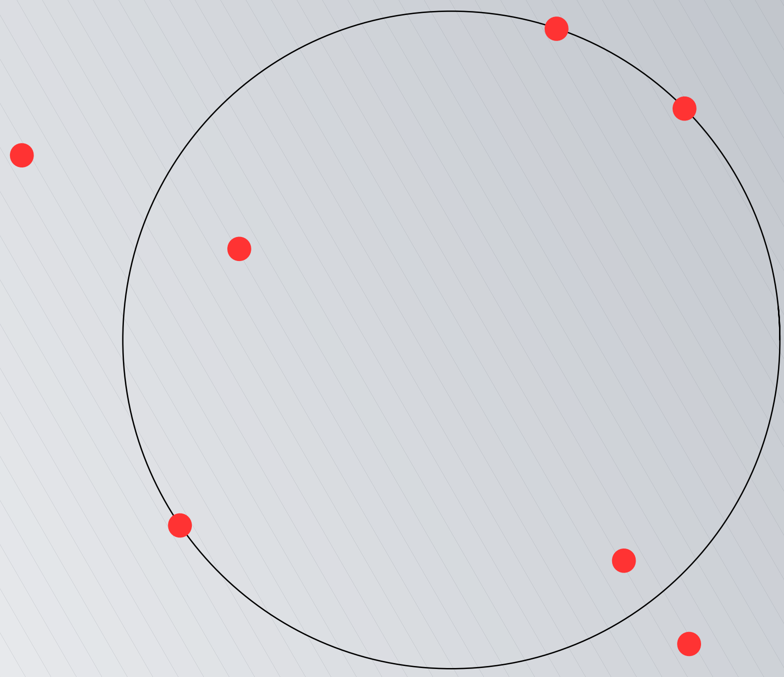
The CR condition implies

$$f(\alpha) = 0 \qquad \Rightarrow f(1/\overline{\alpha}) = 0$$

Moreover $\alpha = 1/\overline{\alpha}$ if and only if $\alpha \in \mathbb{T}$.

Thus the roots of f are either on $\mathbb{T}$ or come is pairs invariant under inversion across $\mathbb{T}$ (hence the term self-inversive).

Definitions



A plot of the roots of

$$x^7 + (\tfrac{1}{2} - \tfrac{i}{3})x^6 - (\tfrac{1}{3} - \tfrac{i}{2})x^5 + \tfrac{3}{2}x^4 + \tfrac{3}{2}x^3 - (\tfrac{1}{3} + \tfrac{i}{2})x^2 + (\tfrac{1}{2} + \tfrac{i}{3})x + 1$$

More Definitions

We will be primarily interested in monic CR polynomials. That is, polynomials of the form

$$f(x) = x^N + 1 + \sum_{n=1}^{N-1} c_n x^{N-n} \qquad c_{N-n} = \overline{c_n}$$

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$$c_{N/2} = \overline{c_{N/2}} \qquad \Rightarrow \qquad c_{N/2} \in \mathbb{R}.$$

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We may identify the set of monic CR polynomials of degree N with $\mathbb{R}^{N-1}$.

Yet More Definitions

For example, we identify $\mathbf{a} \in \mathbb{R}^2$ with the degree 3 CR polynomial

$$\mathbf{a}(x) = x^3 + (a_1 + a_2 i)x^2 + (a_1 - a_2 i)x + 1.$$

and, if $\mathbf{a} \in \mathbb{R}^3$ we set

$$\mathbf{a}(x) = x^4 + (a_1 + a_3 i)x^3 + a_2 x^2 + (a_1 - a_3 i)x + 1.$$

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For example, we identify $\mathbf{a} \in \mathbb{R}^2$ with the degree 3 CR polynomial

$$\mathbf{a}(x) = x^3 + \frac{\sqrt{2}}{2}(a_1 + a_2 i)x^2 + \frac{\sqrt{2}}{2}(a_1 - a_2 i)x + 1.$$

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We do this so that the p-norm on the coefficients of $\mathbf{a}(x)$ is equal to 2 plus the p-norm on the vector $\mathbf{a}$.

The Definition of W_N

Finally we define,

$$W_N = \{\mathbf{a} \in \mathbb{R}^{N-1} : \mathbf{a}(x) \text{ has all roots on } \mathbb{T}\}.$$

That is, W_N is the collection of vectors in $\mathbb{R}^{N-1}$ which correspond to degree N CR polynomials with all roots on the unit circle.

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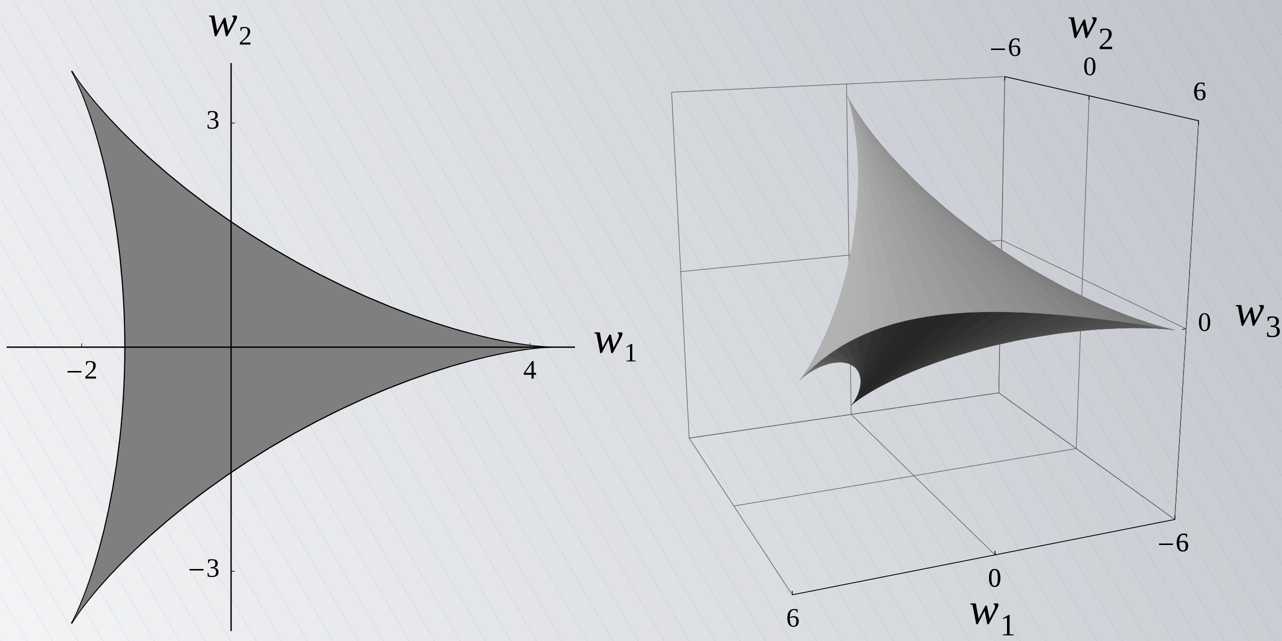

Plots of W_3 and W_4.

Theorem 1. W_N *is homeomorphic to the closed* $N-1$ *dimensional ball,* B^{N-1}*.*

Theorems

Theorem 2. *W_N is homeomorphic to the closed $N-1$ dimensional ball, B^{N-1}.*

Given $\mathbf{a} \in W_N$ there is a natural way to define a (cyclically ordered) partition of N, $\mathcal{P}(\mathbf{a})$ corresponding to the multiplicities of the cyclically ordered roots of $\mathbf{a}$

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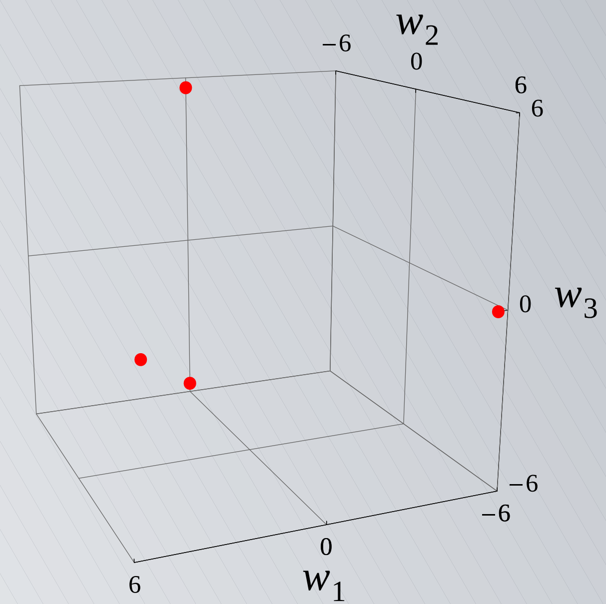
Points in W_4 corresponding to the partition $\{4\}$.



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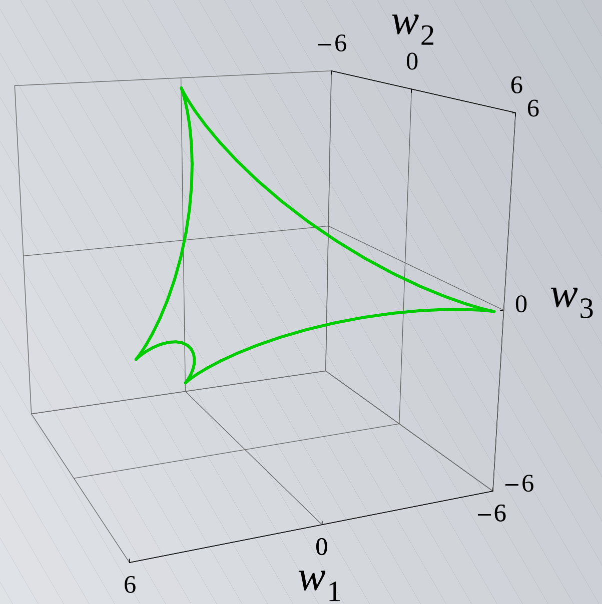
Points in W_4 corresponding to the partition $\{3, 1\}$.



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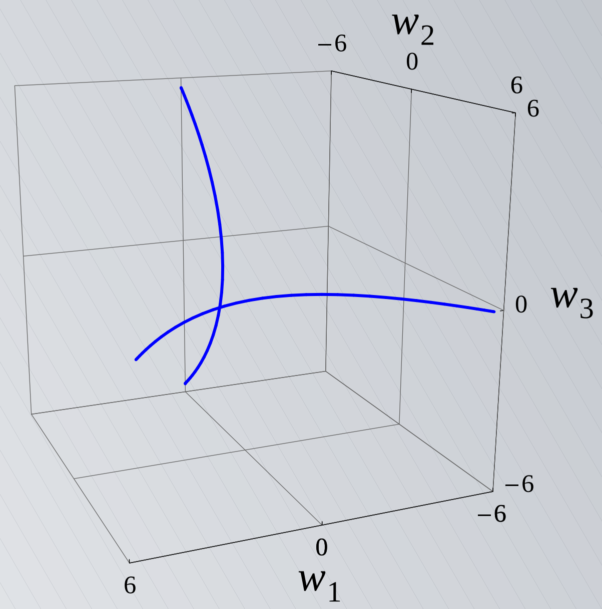
Points in W_4 corresponding to the partition $\{2, 2\}$.



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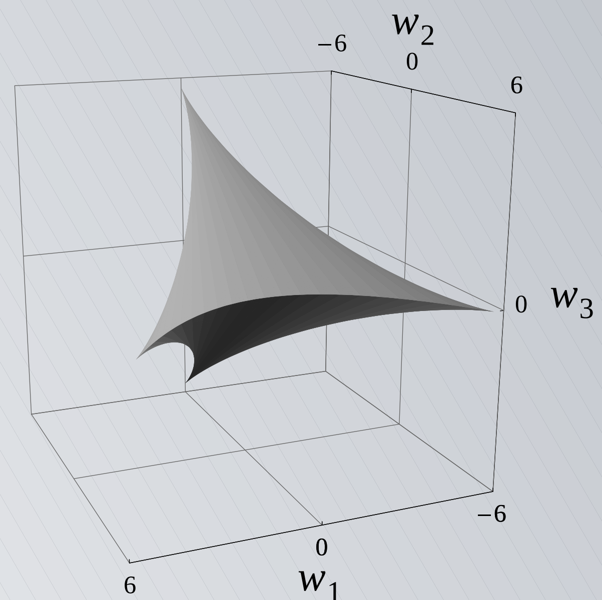
Points in W_4 corresponding to the partition $\{1, 1, 1, 1\}$.



W_4 *Coloured*

$\{4\}$,
$\{3, 1\}$,
$\{2, 2\}$,
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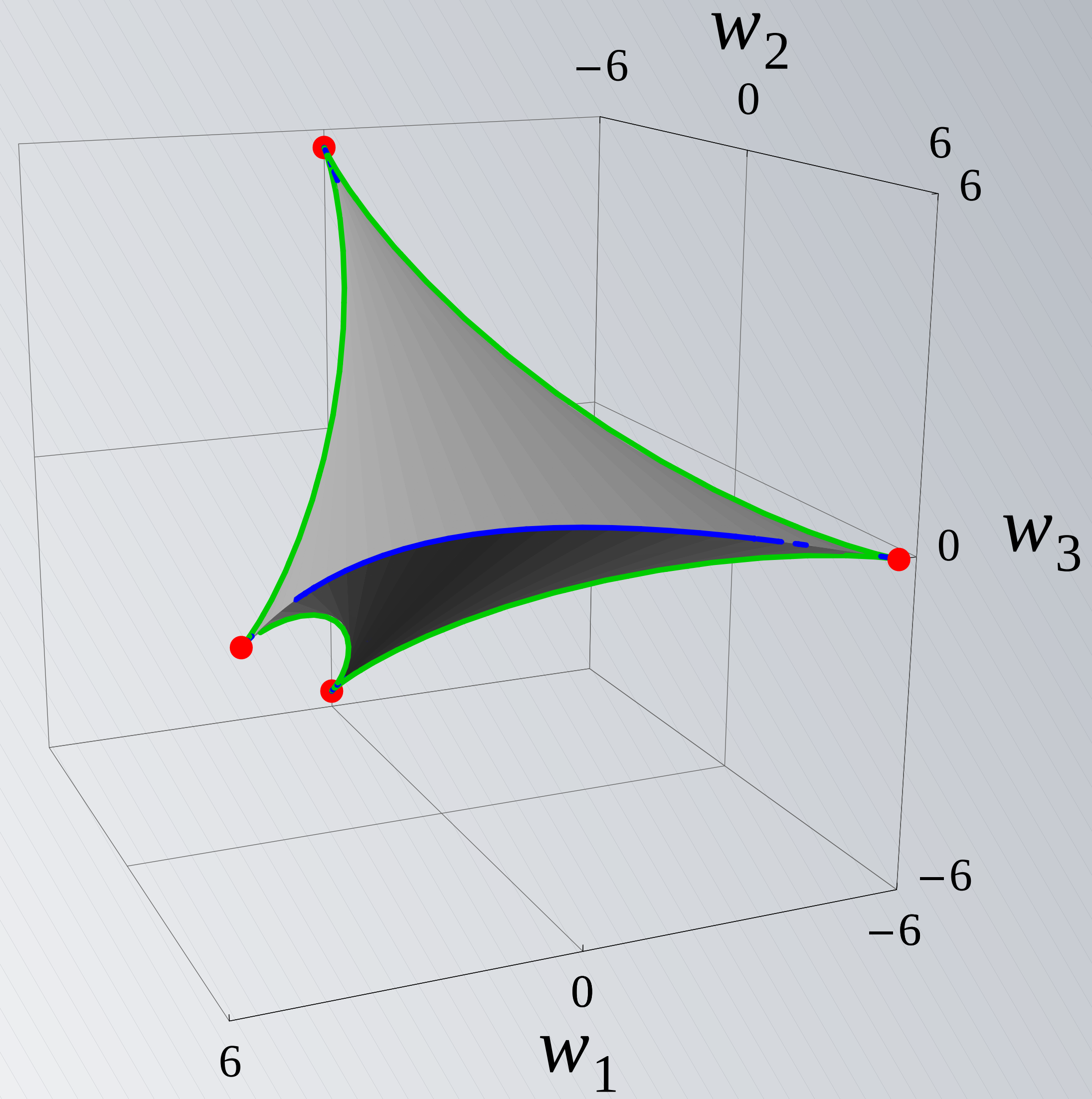

Theorem 3. *The group of isometries of W_N is isomorphic to D_N, the dihedral group of order $2N$.*

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$\mathrm{Isom}(W_N)$ is generated by the isometries R and C where R corresponds to the action of multiplying each root by a primitive N-th root of unity, and C corresponds to complex conjugation of the roots.

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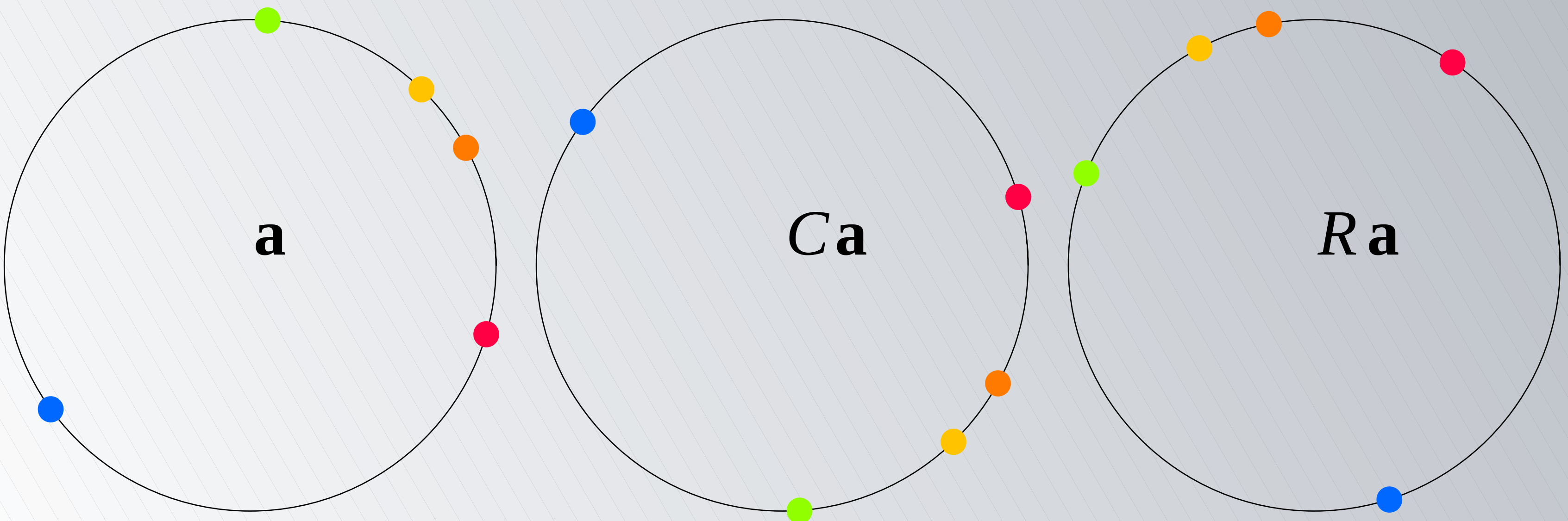

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$$\mathrm{vol}(W_N) = \frac{2^{N-1}\pi^{(N-1)/2}}{\Gamma(\frac{N+1}{2})}.$$

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Theorem 4. *The volume (Lebesgue measure) of W_N is given by*

$$\mathrm{vol}(W_N) = \frac{2^{N-1}\pi^{(N-1)/2}}{\Gamma(\frac{N+1}{2})}.$$

That is, the volume of W_N is equal to the volume of the $N-1$ dimensional ball of radius 2.

Let $B_p^{N-1}(r)$ be the unit ball of radius r with respect to the p-norm on $\mathbb{R}^{N-1}$. Then

Theorem 5. *Let*

$$r_p = \left[\frac{2^p}{(N-1)^{p-1}}\right]^{1/p}$$

Then $B_p^{N-1}(r) \subseteq W_N$ if and only if $r \leq r_p$.

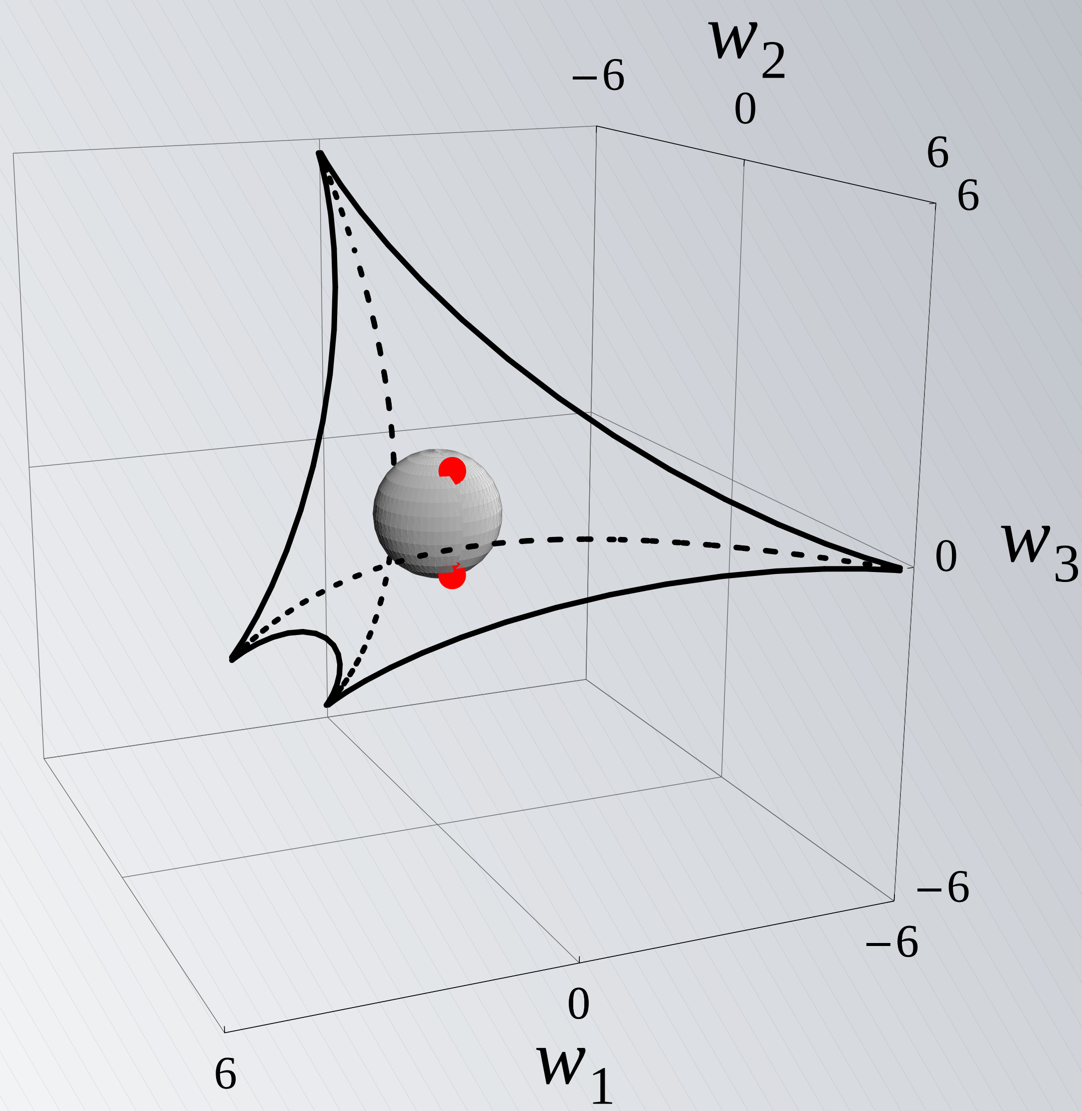
Corollary 7. *Let f be a monic CR polynomial of degree N with a root of multiplicity at least 2. Then,*

$$|f|_p^p \geq 2 + \frac{2^p}{(N-1)^{p-1}},$$

with equality if and only if $f(x) = u(\zeta_N^n x)$ where $\zeta_N = e^{2\pi i n/N}$ and

$$u(x) = x^N - \frac{2}{N-1}x^{N-1} - \frac{2}{N-1}x^{N-2} - \cdots - \frac{2}{N-1}x + 1.$$

Theorems



Corollary 8. *If f is a monic CR polynomial of degree N and*

$$|f|_p^p \le 2 + \frac{2^p}{(N-1)^{p-1}},$$

then f has all roots on the unit circle.

Corollary 9. *If* f *is a monic CR polynomial of degree* N *and*

$$|f|_p^p \leq 2 + \frac{2^p}{(N-1)^{p-1}},$$

then f *has all roots on the unit circle.*

Looking at the proof of this fact, one sees

Theorem 9. *If* f *is a monic CR polynomial with* $L+2$ *non-zero coefficients and*

$$|f|_p^p \leq 2 + \frac{2^p}{L^{p-1}},$$

then f *has all roots on the unit circle.*

Theorem 10. *Suppose $N = 2M$, and let $\mathbf{u} \in \mathbb{R}^{N-1}$ be given by*

$$u_m = \begin{cases} -\frac{2\sqrt{2}}{N-1} & m < M \\ -\frac{2}{N-1} & m = M \\ 0 & \textit{otherwise.} \end{cases}$$

That is,

$$\mathbf{u}(x) = x^N - \frac{2}{N-1}x^{N-1} - \frac{2}{N-1}x^{N-2} - \cdots - \frac{2}{N-1}x + 1.$$

$$\textit{If} \qquad \mathbf{w} \cdot R^n \mathbf{u} \leq \frac{4}{N-1} \qquad n = 1, \ldots, N,$$

then $\mathbf{w}(x)$ has all roots on the unit circle.

Corollary 10. *Suppose $N = 2M$, and let $\mathbf{t} \in \mathbb{R}^{N-1}$ be given by*

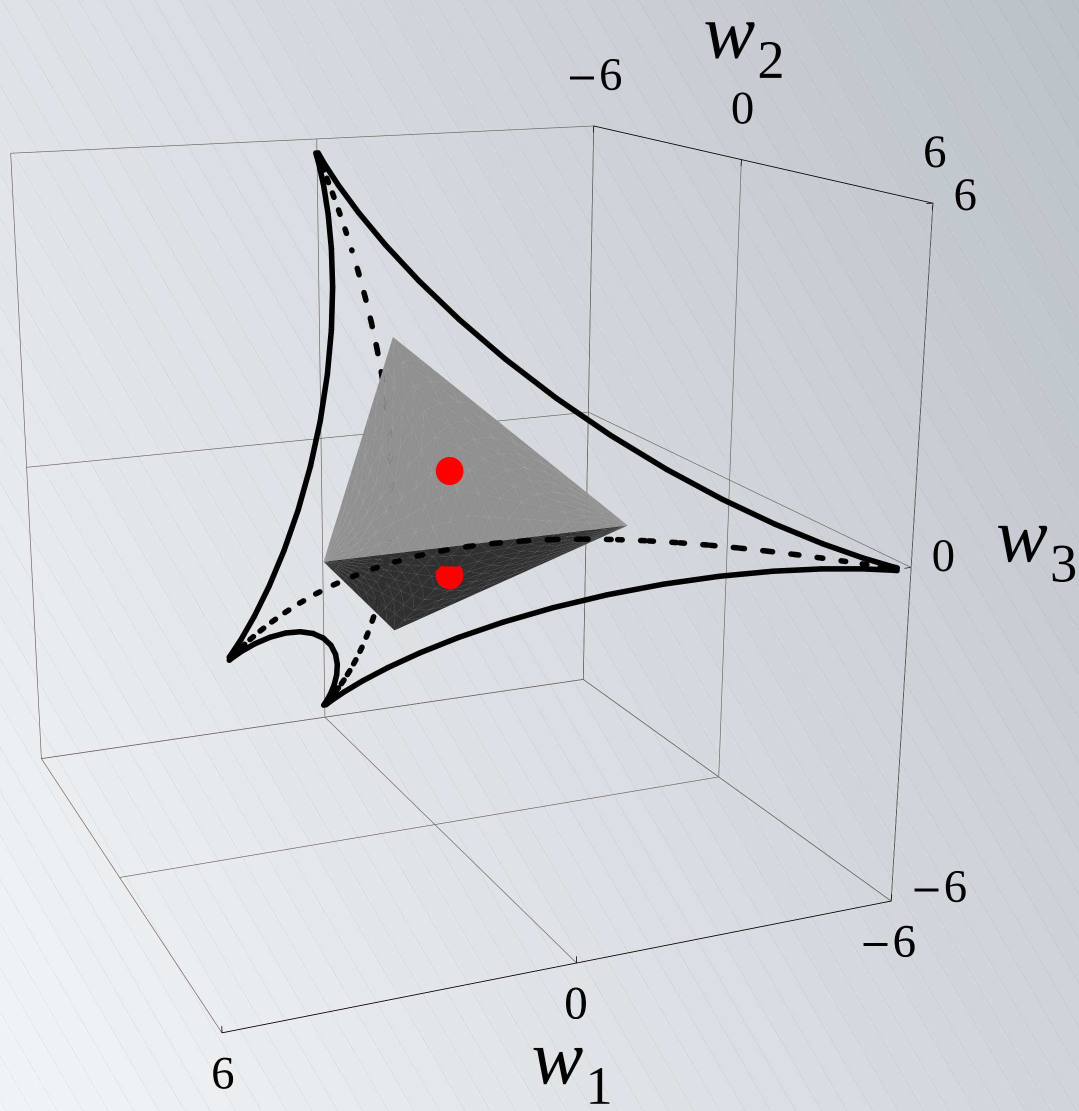
$$t_m = \begin{cases} 2\sqrt{2} & m < M \\ 2 & m = M \\ 0 & \text{otherwise.} \end{cases}$$

That is,

$$\mathbf{t}(x) = x^N + 2x^{N-1} + 2x^{N-2} + \cdots + 2x + 1.$$

If $\mathbf{w}$ is a convex linear combination of $\{R^n\mathbf{t} : n = 1, \ldots, N\}$ then $\mathbf{w}(x)$ has all roots on the unit circle.

Theorems



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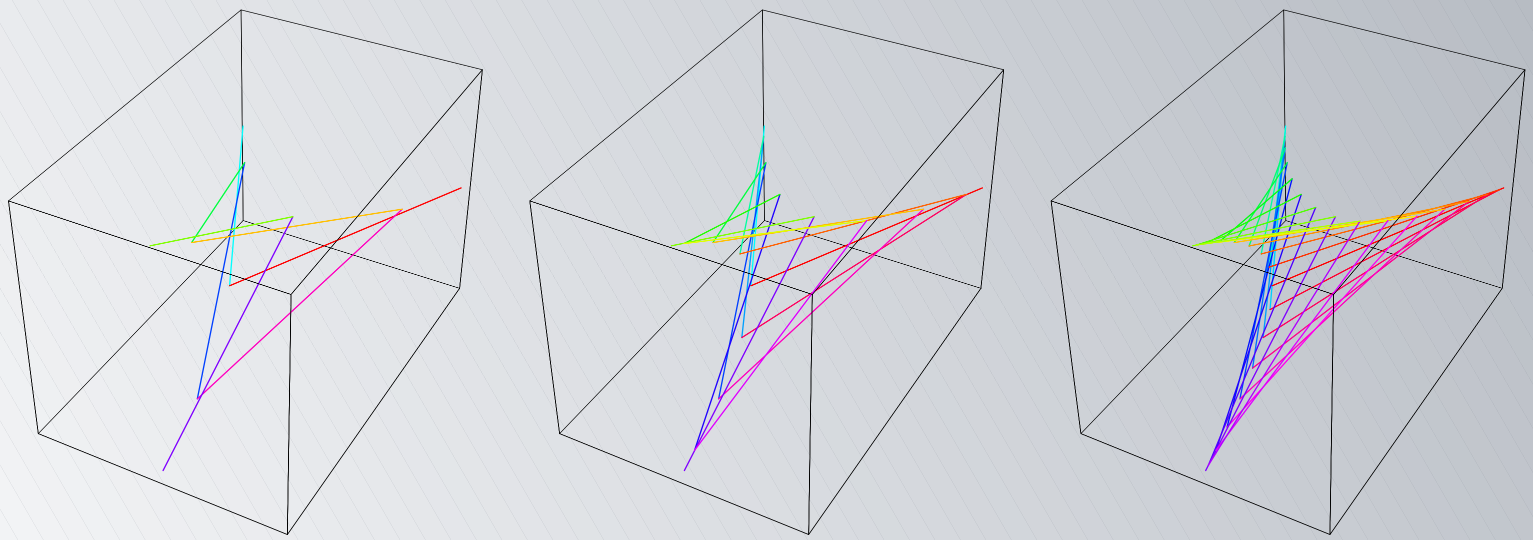

A Few Preliminaries

Proposition 11. *The vector* $\mathbf{w}$ *is in* W_N *if and only if*

$$\mathbf{w}(x) = \prod_{n=1}^{N} (x - \xi_n),$$

where $\xi_1, \xi_2, \ldots, \xi_N$ *are elements of* $\mathbb{T}$ *satisfying* $\xi_1 \xi_2 \cdots \xi_N = (-1)^N$*.*

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Proof.

$$x^N \overline{\mathbf{w}(1/\overline{x})} = x^N \prod_{n=1}^{N} \left(\frac{1 - \overline{\xi_n} x}{x} \right)$$

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Proof.

$$x^N \overline{\mathbf{w}(1/\overline{x})} = \mathbf{w}(x).$$

It follows that $\mathbf{w}$ is CR and in W_N. The converse is obvious since every element of W_N is a polynomial with all roots on the unit circle and constant coefficient 1.

A Few More Preliminaries

Proposition 12. *The points in W_N corresponding to the partition $\{N\}$ are given by*

$$\mathbf{v}_n(x) = (x + \zeta_N^n)^N \qquad n = 1, 2, \ldots, N,$$

where $\zeta_N = e^{2\pi i/N}$.

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Proposition 13. *The points in W_N corresponding to the partition $\{N\}$ are given by*

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We call the $\mathbf{v}_n$ the *vertices* of W_N.

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Corollary 13. *Each Isometry of W_N must map the set of vertices to itself. That is, $\mathrm{Isom}(W_N)$ is a subgroup of S_N.*

The Distance Between Vertices

Lemma 14. *For each* $1 \leq M \leq N$*, the vertices farthest from* $\mathbf{v}_M$ *are* $\mathbf{v}_{M+1}$ *and* $\mathbf{v}_{M-1}$*.*

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This lemma implies that $\mathrm{Isom}(W_N) \equiv D_N$.

Lemma 14. *For each $1 \leq M \leq N$, the vertices farthest from $\mathbf{v}_M$ are $\mathbf{v}_{M+1}$ and $\mathbf{v}_{M-1}$.*

- Given $T \in \mathrm{Isom}(W_N)$, there exists an L such that $(R^L T)\mathbf{v}_N = \mathbf{v}_N$.

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- C maps $\mathbf{v}_{N-1}$ to $\mathbf{v}_1$.
- There exists $K \in \{0, 1\}$ such that $S := C^K R^L T$ fixes both $\mathbf{v}_N$ and $\mathbf{v}_1$.

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Self Inversive Polynomials

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As in the CR case, the set of monic ω-CR polynomials is in bijective correspondence with $\mathbb{R}^{N-1}$.

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We define

$$W_{N,\omega} = \left\{ \mathbf{a} \in \mathbb{R}^{N-1} : (x^N + \omega) + \sum_{n=1}^{N-1} c_n x^{N-n} \text{ has all roots on } \mathbb{T} \right\}$$

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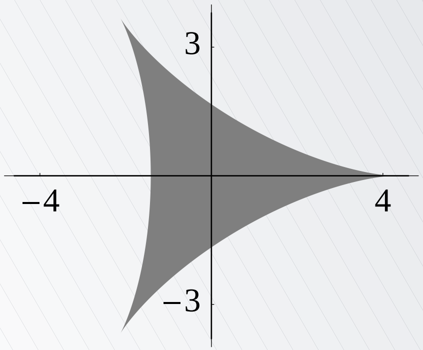
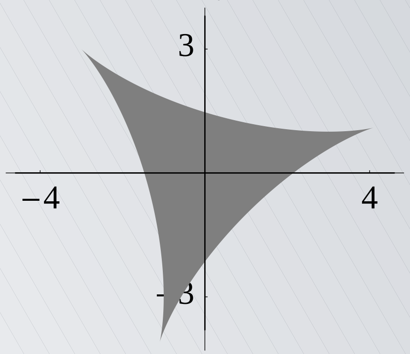
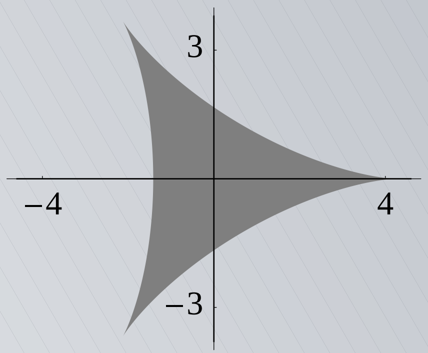
$W_{3,1}$



$W_{3,\zeta_{16}}$



$W_{3,-1}$



A Change of Variables

Proposition 14. *The vector* $\mathbf{w}$ *is in* $W_{N,\omega}$ *if and only if*

$$\mathbf{w}(x) = \prod_{n=1}^{N} (x - \xi_n),$$

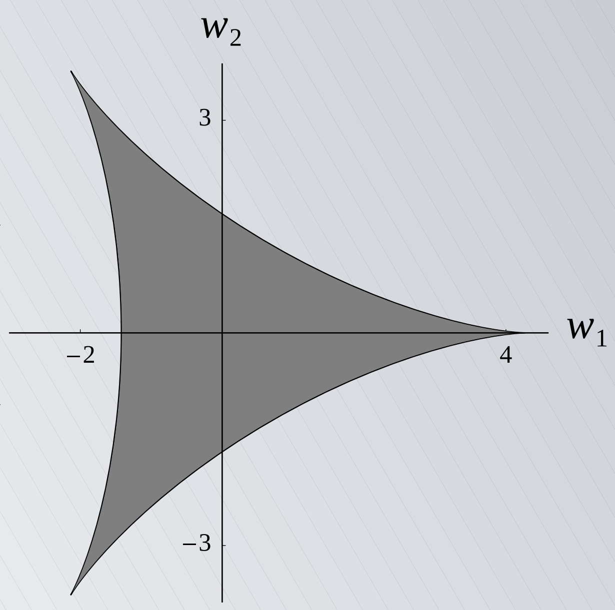
where $\xi_1, \xi_2, \ldots, \xi_N$ *are elements of* $\mathbb{T}$ *satisfying* $\xi_1 \xi_2 \cdots \xi_N = (-1)^N \omega$*.*

We define the map $E_{N,\omega} : \mathbb{T}^{N-1} \to W_{N,\omega}$ specified by $\mathbf{a} = E_{N,\omega}(\boldsymbol{\xi}) = X_{N,\omega}^{-1}\mathbf{c}$ where $\mathbf{c}$ is obtained from $\boldsymbol{\xi}$ by

$$\left(x - \frac{(-1)^N \omega}{\xi_1 \xi_2 \cdots \xi_{N-1}}\right) \prod_{n=1}^{N-1} (x - \xi_n) = (x^N + \omega) + \sum_{n=1}^{N-1} c_n x^{N-n}.$$

The Volume of W_N

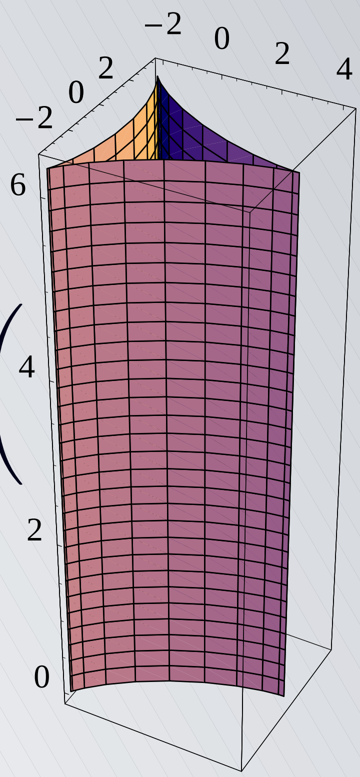
$$\mathrm{vol}(W_N) = \mathrm{vol}\left(\right.$$



$$\left.\right)$$

The Volume of W_N

$$\mathrm{vol}(W_N) = \frac{1}{2\pi}\,\mathrm{vol}\left(\quad \right)$$



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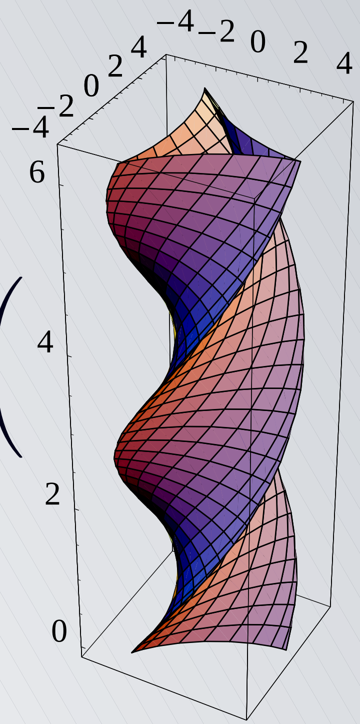

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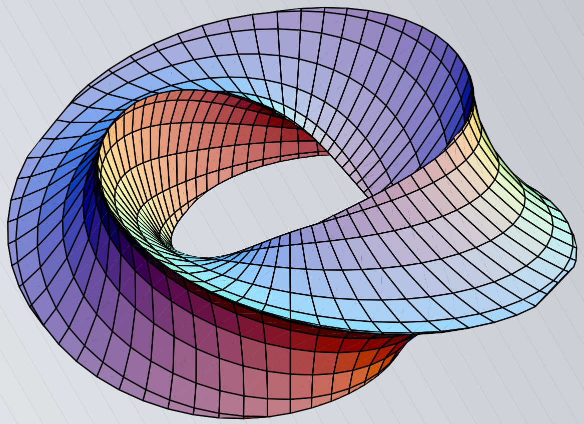

cruller.mov

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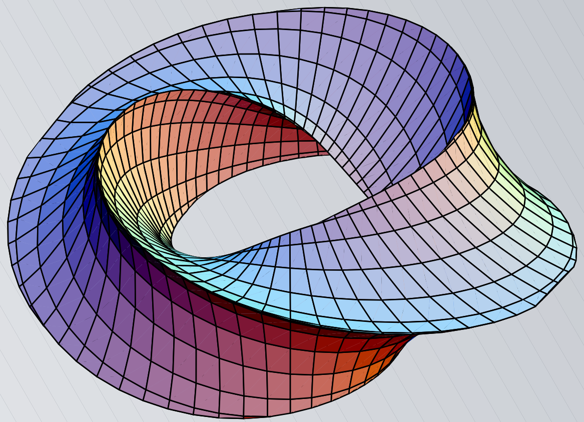
$\mathrm{vol}(W_N) = \frac{1}{2\pi}\,\mathrm{vol}\Bigg($  $\Bigg)$

That is,

$$\mathrm{vol}(W_N) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{vol}(W_{N,e^{i\theta}})\, d\theta$$

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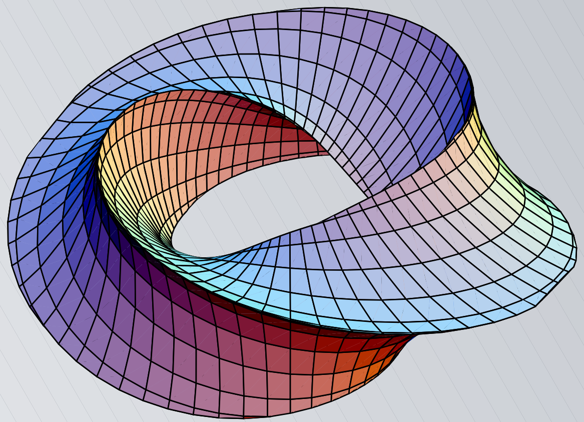
$$\mathrm{vol}(W_N) = \frac{1}{2\pi}\,\mathrm{vol}\Bigg($$  $$\Bigg)$$

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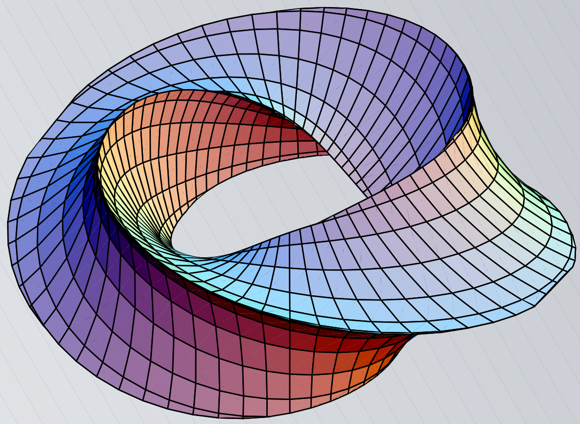

That is,

$$\mathrm{vol}(W_N) = \frac{1}{2\pi N!}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{1\le m<n\le N}\left|e^{i\theta_n}-e^{i\theta_m}\right|d\theta_1\cdots d\theta_N.$$

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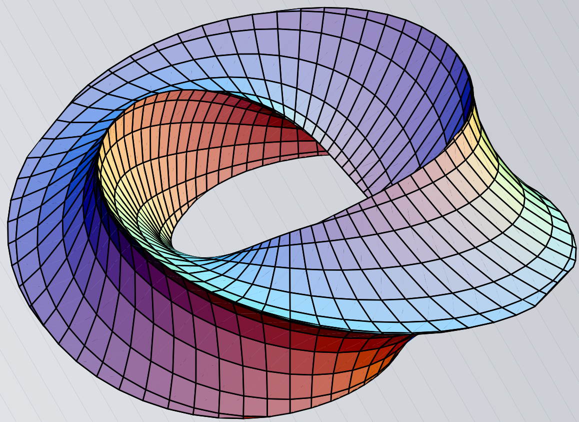
$$\mathrm{vol}(W_N) = \frac{1}{2\pi}\,\mathrm{vol}\Bigg(\quad$$  $$\Bigg)$$

Dyson computed this integral in the context of RMT, and

$$\mathrm{vol}(W_N) = \frac{2^{N-1}\pi^{(N-1)/2}}{\Gamma\left(\frac{N+1}{2}\right)}.$$