Koszul duality 2

Consider an abelian category $\mathcal{C}$, and an object L.

What does the category $\mathcal{C}$ have to do the Ext algebra $B = \mathrm{Ext}^\circ(L,L)$?

If L is projective, then the answer is Morita theory:

We have functors

$$\mathcal{C} \underset{N\otimes_B L \leftarrow\!\shortmid N}{\overset{M \mapsto \mathrm{Ext}(L,M)}{\rightleftarrows}} \mathrm{Mod}\text{-}B$$

The composition is always an equivalence.

The composition is an equivalence on the subcategory of L-presentable objects (the ones w/ a presentation of the form $0 \leftarrow M \leftarrow L\otimes V \leftarrow L \otimes W \leftarrow 0$. This corresponds to the presentation

$$0 \leftarrow \mathrm{Ext}(L,M) \leftarrow B\otimes V \leftarrow B \otimes W \leftarrow)$$

Thus, these are equivalences iff every object in $\mathcal{C}$ is L-presentable

What if L is not projective?
then
We still have $\mathrm{Ext}(L,M)$ which is now a graded R-module. But how are we supposed to make sense of $\mathrm{Ext}(L,M) \otimes L$?

Clearly just thinking of a graded module isn't enough. Instead we need think of $\mathrm{Ext}(L,L)$ as a dg-algebra, and $\mathrm{Ext}(L,M)$ as a dg-module.

Thus, we don't think of $\mathrm{Ext}(L,L)$ as a graded vector space, but pick a projective res. $P^\bullet \to L$ and consider $\bigoplus \mathrm{Hom}(P^i, P^k)$ w/ grading $i-k$, and differential $f \mapsto \partial f - (-1)^{i-k} f \partial$.

Similarly by $\mathrm{Ext}(L,M) =: E$ I mean the module $\bigoplus \mathrm{Hom}(P^i, M)$ with the grading i and differential $f \mapsto (-1)^i f \partial$.

Now, I know what tensor product means: the complex

$$\bigoplus E \otimes L \oplus E \otimes B \otimes L^{[1]} \oplus E \otimes B \otimes B \otimes L^{[2]} \oplus \cdots$$

$\partial(e \otimes b_1 \otimes \cdots \otimes b_m \otimes l) = \pm\, de \otimes \cdots \otimes l$
$\pm\, e \otimes \partial b_1 \otimes \cdots \otimes b_m \otimes l + \cdots +$
$e \otimes b_1 \otimes \cdots \otimes \partial l \pm e b_1 \otimes \cdots \otimes l$
$\pm\, e \otimes b_1 b_2 \otimes \cdots \otimes l \pm e \otimes \cdots \otimes b_m l$

$e \otimes b \otimes l$ imposes the relation
$(eb) \otimes l = e \otimes bl$.

Note: I had been assuming L was an object, but I could equally well take it to be a complex (which is why I included the differential above).

Essentially automatically, we get an (quasi)equivalence between the dg-category generated by L and the dg category of B-modules

A "free" resolution of Ext(L, M) is the "instructions" for building ~~an~~ a complex out of copies of L.

Ok, what if $\underset{=\oplus L_i}{L}$ is semisimple?

~~dg-category of (projective) complexes with cohomology (having) all~~ ~~L's~~

Let $\mathcal{C}_L$ be subcategory w/ all composition factors ~~than~~ $\cong L_i$

Let $D(\mathcal{C})_L$ be dg-category of projective complexes w/ cohomology in $\mathcal{C}_L$.

Thm $D(\mathcal{C})_L \cong$ dgmod-$\mathrm{Ext}(L,L)$

(Assuming $\mathcal{C}$ has all limits)

Objects with bounded, finite length cohomology $\iff$ perfect dg-modules. (those with "finite projective resolutions").

~~Obviously~~ So, this means that the objects in $\mathcal{C}_L$ itself turn into dg-modules. Presumably, these actually have special properties.

$\mathrm{Ext}(L_k, L_i) \cong e_i B$ for $e_i : L \to L_i \to L$

If M is an extension of these two modules, then L_K, L_i then we have

$$\begin{tikzcd}
0 \arrow[r] & L_K \arrow[r] & M \arrow[r] & L_i \arrow[r] & 0 \\
& P^0_K \arrow[u] & P^0_K \oplus P^0_i & P^0_i & \\
& P^1_K \arrow[u] & P^1_K \oplus P^1_i \arrow[u] & P^1_i \arrow[u] & \\
& P^2_K \arrow[u] & P^2_K \oplus P^2_i \arrow[u] & P^2_K \arrow[u] &
\end{tikzcd}$$

So more generally, we have that $M \in \mathcal{C}_L$ is going to seem to some sort of amalgamation of the modules $e_i B_q$. What this means is kind of tricky in general.

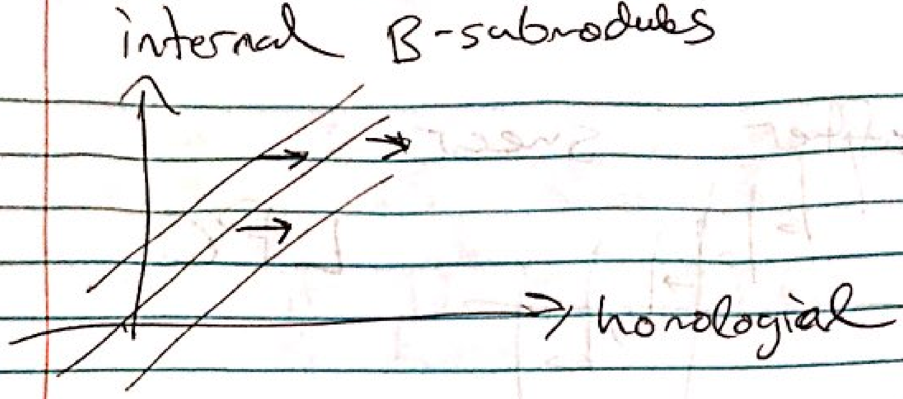
Assume that $\mathcal{C}$ has a graded lift $\tilde{\mathcal{C}}$, with L gradable. Also assume that the internal degree on $\mathrm{Ext}(L,L)$ agrees with homological degree.

(Note: this happens if $L = A/A_{>0}$ for A Koszul, but is more general; ~~It~~ It could be some subset of simples for a Koszul algebra, for example).

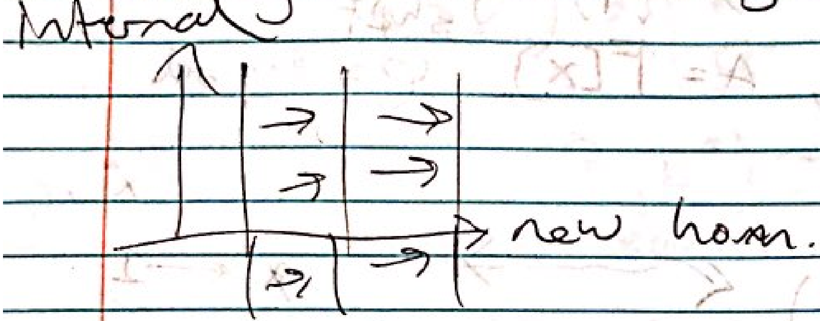
In this case, any $M \in \mathcal{C}_L$ has a projective resolution gotten by gluing resolutions of L. The grading condition implies that ~~Ext~~ B is formal (i.e. after a quasi-isomorphism, it has trivial differential).

Thus $\mathrm{Ext}(L,M)^{\mathbb{Z}}$ is a sum of $e_i B$'s as honest modules, with an additional differential.

This differential ~~that~~ increases homological degree and leaves internal degree unchanged, whereas B keeps the difference constant



If we tilt, ~~and~~ Keeping internal degree the same, and making new hom deg. = hom − internal.



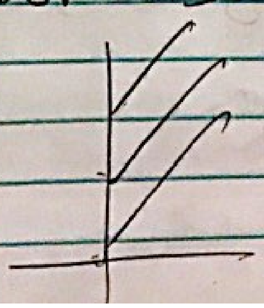
Thus, we get a complex of graded B-modules (not dg; honest). dg just means we still only think up to quasi-isomorphism.

This argument works for any object of $D(\mathcal{C}_2)$, Actually we get an ~~isomorphism~~ quasi equivalence

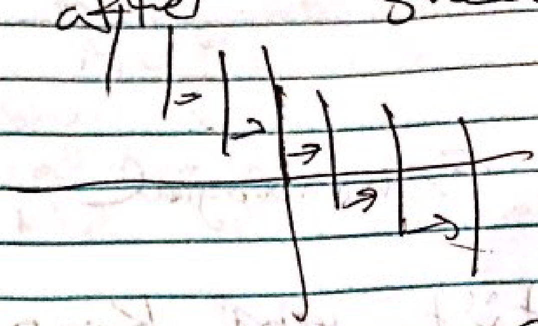
$$D(\mathcal{C}_2) \simeq D^+(B\text{-grmod})$$

(modulo issues about finiteness)

What's special about $\mathcal{C}_2$? generated in homological degree 0.



so, after shear.



LPC!

Thm $\tilde{\mathcal{C}}_L \cong$ perfect LPCs over B.

graded shift $\langle i \rangle \longleftrightarrow \{i\}$ grading and homological shift

For example, $A = F[x]$, $\mathcal{C} = A$-mod.

$B = \Lambda(x^*)$ $\qquad x^*$,

$F[x]/(x^n) \longleftrightarrow \quad x^* \leftarrow 1$

$x^* \leftarrow 1$

$\mathrm{Ext}(F, F[x]/(x^n))$ $\quad 1$ of F

$\cong H^*(F[x]/(x^n) \to F[x]/(x^n))$

$0 \quad F$

$0 \quad 0$

$F \quad 0$

$0 \quad 0$

$0 \quad 0$

$F \quad 0$

related by shear. (cohomology of the complex)

Cor $\tilde{\mathcal{C}}_L \cong B^!$-gmod.

Note! $D^b(\tilde{\mathcal{C}}_L) \longrightarrow D^b(B\text{-gmod})$

$\cong D^b(B^!\text{-gmod})$

$M \longmapsto M \otimes_{B^!} K^\bullet$

When is this functor an equivalence?
Thm (MOS) When B is Koszul.

Cor if $D^b(\mathcal{C}_2) \cong D(\mathcal{C})_2$, then B is Koszul.

Key observation: in $LP(B)$, K is ~~Koszul~~ injective. Thus

$$\mathrm{Ext}^i_{LP}(P_i, K^\bullet) = \begin{cases} F & i=0 \\ 0 & i\neq 0 \end{cases}$$

However $\mathrm{Ext}^i_B(P_i, K^\bullet) \cong H^i(K^\bullet)e_i$
thus equivalence requires exact Koszul.

Finally, we've talked about highest weight categories

Def We call a category $\mathcal{C}$ with order on its simples standard Koszul if the standard modules have linear projective resolutions, and its costandards have linear injectives.

~~Thm~~

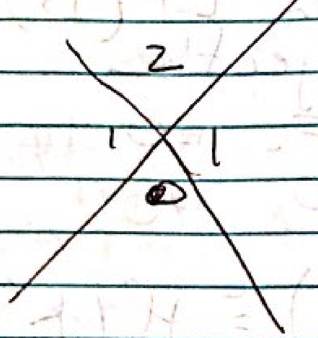
Note if $\mathcal{C} = A$-mod with $A \cong A^{op}$, then taking duals makes these conditions redundant.

Thm If $\mathcal{C}$ is standard Koszul, then it is Koszul in the usual sense and its Koszul dual is also standard Koszul.

Actually if Koszul, HW and KD also HW $\Leftrightarrow$ standard Koszul

Theorem: For any hyperplane arrangement, and choice of η and ξ which is generic, the A-algebra is standard Koszul.

Pf



The projectives in the lines are indexed by the chambers, recording the maximum of the chamber

The differentials are all step across the wall, with a few signs added.

Exactness is basically inclusion-exclusion: every path can be factored as going up and then going down. If the path goes up at all, it's in the image of the ~~length~~ first step: if it's not there, then in the second step it appears to cancel out the extras, etc.