Introduction to symplectic duality (the abelian case)

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For a lot of history, it seemed as though commutative rings were maybe the most natural framework in which to view mathematics.

- Obvious context for number theory, algebraic geometry.
- In physics, observable quantities form a commutative ring.

Then quantum mechanics came along, and the picture looked a bit different. The algebra of observables becomes non-commutative, but with a classical limit (it's "**almost commutative**").

On the classical side, physicists had already noticed a hint of the non-commutativity of quantum mechanics: the Poisson bracket (which is often called "**semi-classical**")

Hamilton's equation (for an observable): $$\frac{\partial f}{\partial t} = \{H, f\}$$

Heisenberg's equation (for an operator): $$i\hbar\frac{\partial \hat{f}}{\partial t} = [\hat{H}, \hat{f}]$$

commutative

non-commutative

commutative semi-classical almost commutative non-commutative

$\underbrace{\text{commutative} \qquad \text{semi-classical}}_{\text{algebraic geometry}} \qquad \underbrace{\text{almost commutative} \qquad \text{non-commutative}}_{\text{representation theory}}$

$$\overbrace{\text{commutative} \qquad\qquad \text{semi-classical}}^{\text{algebraic geometry}} \quad \text{X} \quad \overbrace{\text{almost commutative} \qquad \text{non-commutative}}^{\text{representation theory}}$$

$\uparrow$

where I'd like to be

Definition

An **almost commutative ring** *is a ring A with a filtration $A_0 \subset A_1 \subset \cdots$ and an integer $n > 0$ such that*

$$A_iA_j \subset A_{i+j} \qquad [A_i, A_j] \subset A_{i+j-n}$$

In particular, the ring $\mathrm{gr}(A) \cong \oplus_{i=0}^{\infty} A_i/A_{i-1}$ is commutative and $\mathbb{Z}_{\geq 0}$-graded.

The ring $\mathrm{gr}(A)$ inherits a semi-classical structure:

Definition

A **conical Poisson ring** *is a $\mathbb{Z}_{\geq 0}$-graded commutative ring R with a second operation $\{-,-\}\colon R \times R \to R$, homogeneous of degree $-n$, that satisfies the relations of a Lie bracket (bilinear, anti-symmetric, Jacobi) such that the Leibniz rule holds:*

$$\{ab, c\} = a\{b, c\} + b\{a, c\}.$$

There's a **classical limit** functor $A \mapsto (\mathrm{gr}(A), \{-,-\})$ from almost commutative algebras to conical Poisson algebras, with the Poisson bracket given by

$$\{\bar{a}, \bar{b}\} = \overline{[a, b]} \in A_{i+j-n}/A_{i+j-n-1}.$$

The most basic case is when $A = \mathbb{C}\langle x, \frac{d}{dx}\rangle$ is the algebra of polynomial differential operators. This is filtered with

$$A_1 = \operatorname{span}\left(1, x, \frac{d}{dx}\right) \qquad A_n = A_1^n$$

This is almost commutative ($n = 2$) with $\operatorname{gr}(A) = \mathbb{C}[x, p]$.

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial x} \qquad \{p, x\} = 1$$

Similarly, $U(\mathfrak{g})$ for any Lie algebra $\mathfrak{g}$ is almost commutative, with classical limit $\mathbb{C}[\mathfrak{g}^*]$ with the KKS Poisson structure.

Definition

If R is an conical Poisson algebra, then a **quantization** *of R is an almost commutative algebra A whose classical limit is R.*

You can easily check that A is the unique quantization of $\mathbb{C}[x,p]$. (Hint: $A_1 \cong \mathbb{C} \cdot \{x, p, 1\}$ as 3-dimensional Lie algebras).

What happens when we consider other Poisson varieties?

In general, finding all quantizations is not easy; Kontsevich got a Fields Medal in large part for doing so for a *real* Poisson structure on $\mathbb{R}^n$.

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We call an affine variety Y conical Poisson if its coordinate ring has that structure.

Definition

We call Y a **conical symplectic variety** *(i.e. conical variety w/ symplectic singularities) if the Poisson bracket induces a symplectic structure on the smooth locus (+silly technical conditions).*

- If $\mathbb{C}^{2n}$ has the usual symplectic structure, and Γ is a finite group preserving ω, then $Y \cong \mathbb{C}^{2n}/\Gamma$ is an example.
- The variety of nilpotent matrices (and more generally, nilpotent cone of a semi-simple Lie algebra $\mathfrak{g}$) has a natural symplectic structure.

The correspondence between almost commutative and semi-classical is particularly nice in this case.

Theorem (Namikawa)

Every conic symplectic variety Y has a universal deformation $\mathcal{Y}$ that smoothes it out as much as possible while staying symplectic (which is thus rigid). The base of this deformation is a vector space H.

For example, $\mathfrak{g}^*$ comes from the nilcone $\mathcal{N}$. The simplest example is:

$$Y = \mathbb{C}^2/(\mathbb{Z}/\ell\mathbb{Z}) \cong \{(u = x^\ell, v = y^\ell, w = xy) \mid uv = w^\ell\}$$

The universal deformation is given by adding formal coordinates a_i on H of degree $2i$.

For a general $\mathbb{C}^{2n}/\Gamma$, there's a similar deformation direction attached to every conjugacy class of symplectic reflection (an element that fixes a codimension 2 subspace).

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For example, $\mathfrak{g}^*$ comes from the nilcone $\mathcal{N}$. The simplest example is:

$$\mathcal{Y} \cong \{(u, v, w, a_1, \dots a_\ell) \mid uv = w^\ell + a_1 w^{\ell-1} + \cdots + a_\ell\}$$

The universal deformation is given by adding formal coordinates a_i on H of degree $2i$.

For a general $\mathbb{C}^{2n}/\Gamma$, there's a similar deformation direction attached to every conjugacy class of symplectic reflection (an element that fixes a codimension 2 subspace).

So, how do we get an analogue of the universal enveloping algebra?

Theorem (Bezrukavnikov-Kaledin, Braden-Proudfoot-W., Losev)

Every conical symplectic variety Y has a unique quantization A of its universal Poisson deformation $\mathcal{Y}$; this is an almost commutative algebra such that $\operatorname{gr} A \cong \mathbb{C}[\mathcal{Y}]$.

The center $Z(A)$ is the polynomial ring $\mathbb{C}[H]$; the quotient A_λ for a maximal ideal $\lambda \in H$ has an isomorphism $\operatorname{gr} A_\lambda \cong \mathbb{C}[Y]$. This gives a complete irredundant list of quantizations of $\mathbb{C}[Y]$.

- If Y is a nilcone, then A is the universal enveloping algebra.
- If $Y \cong \mathbb{C}^{2n}/\Gamma$, then A is a spherical symplectic reflection algebra.

This workshop will be focused on another example of a symplectic singularity: **hypertoric varieties**.

Let K be a connected subgroup of diagonal matrices on $\mathbb{C}^n$. Consider $T^*\mathbb{C}^n$ with the induced K action.

Definition

The **universal hypertoric variety** *for this action is*

$$\mathscr{M} = T^*\mathbb{C}^n /\!\!/ K = \mathrm{Spec}(\mathbb{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]^K).$$

The **hypertoric variety** $\mathfrak{M}$ *is the fiber over 0 of the composed map:*

$$\mathscr{M} \to \mathbb{C}^n \to \mathfrak{k}^*$$
$$(x_1, \dots, \xi_n) \mapsto (x_1\xi_1, \dots, x_n\xi_n)|_{\mathfrak{k}}$$

This is the moment map for the action of K.

Proposition

The variety $\mathscr{M}$ is conical Poisson, inheriting the Poisson bracket for the usual one:

$$\{x_i, \xi_j\} = \delta_{ij} \qquad \{x_i, x_j\} = \{\xi_i, \xi_j\} = 0.$$

The variety is $\mathfrak{M}$ is conical symplectic, and $\mathscr{M}$ is its universal deformation.

The quantization of this variety can be easily constructed by replacing $T^*\mathbb{C}^n$ by differential operators in n-variables; Nick will do this in the next talk.

Symplectic singularities are the Lie algebras of the 21st century. - Okounkov

There's a very interesting interplay between the geometry of $\mathcal{Y}$ and the representation theory of A, modeled on that of $\mathfrak{g}^*$ and $U(\mathfrak{g})$:

Geometry

- orbit method
- geometric quantization
- flag variety and Schubert varieties
- localization, D-modules, intersection cohomology
- support varieties

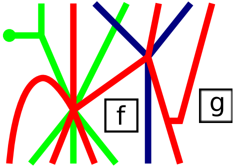
Algebra

- primitive ideals
- Harish-Chandra (bi)modules
- category $\mathscr{O}$
- character formulae
- translation/projective functors

We should really have a third column here:

Combinatorics: Coxeter groups, tableaux, cells, KL polynomials

Culmination is the Soergel calculus of Elias and Williamson.



To algebraists, describes category $\mathscr{O}$ and HC bimodules. To geometers, describes the $B \times B$ equivariant D-modules on G.

The first description is of the endomorphisms of a projective generator, the second is of Ext of a simple generator. This is a relationship we will see many times: **Koszul duality**.

However, if that's the topic you want to hear about, you're 3 years too late.

Kazhdan-Lusztig theory and Soergel bimodules

10 August - 14 August 2015
University of Oregon
Eugene, OR

The goal of this workshop will be to first get a solid handle on Soergel bimodules and the philosophy of algebraic categorification, and then understand the recent proof of the Soergel conjecture in the following paper:

The Hodge theory of Soergel bimodules, by Ben Elias and Geordie Williamson.

The workshop will be led by Ben Elias, and will consist of a combination of lectures and problem sessions. It will be organized by Nicholas Proudfoot and Alexander Ellis.

In this workshop we'll instead consider how these same ideas can be applied to hypertoric varieties.

Why that focus?

1. Much easier: we have a much more hands on understanding of the essentially every aspect of these varieties compared to T^*G/B.
2. Good test case: hypertoric varieties and T^*G/B both have lots of special properties, but in mostly orthogonal ways.
3. Natural connections to physics.

Beginning with a connected reductive complex group G, and a representation V, there's a 3-d $\mathcal{N} = 4$ supersymmetric field theory you can build.

This field theory has a moduli space of vacua (lowest energy states) which is a big reducible algebraic variety with two distinguished components: the **Higgs** and **Coulomb** branches. Both of these are hypertoric varieties if G is a torus.

- The Higgs branch is defined as a symplectic quotient like the definition of the hypertoric variety given before, for the $G = K$ and $V = \mathbb{C}^n$.
- The Coulomb branch has a much more complicated definition we'll discuss this afternoon. It is isomorphic to a quite different hypertoric variety, attached to $T^\vee = \ker\left((\mathbb{C}^*)^n \to K^\vee\right)$.

Central question: how are these related? Physicists call this S-duality; we ended up with the term **symplectic duality.**

Note, by this we mean the connection between Higgs and Coulomb branches in nonabelian case, or even for theories that don't have this form.

Rough plan:

Mon: What are these varieties and their quantizations?

Tues: What are the representations of these quantizations?

Wed: How are the representations of the Higgs and Coulomb branches related (in the abelian case)?

Thurs: How do these relate to the geometry of hypertoric varieties?

Fri: How does this generalize to the nonabelian case?

Today, we'll mostly focus on two definitions of the hypertoric enveloping algebra:

- The Higgs definition is a direct quantization of the definition I wrote: you begin with differential operators, take K invariants, and then, if you choose, you can mod out by a maximal ideal of the center.
- The Coulomb definition might sound confusing at first, but it's physically very natural: it's an algebraic manifestation of the different ways of taking a $T^\vee$ principal bundle with section on a surface, and doing a local modification of it. Convolution with special ways of doing these modifications will match up with special invariant differential operators, and give an isomorphic algebra.

- As time allows, we'll apply these presentations to learn more about the structure of representations of this algebra, using weight theory (which actually much easier in this case than in Lie algebras).

Tomorrow, we'll focus on the category of representations of this algebra, and a special subcategory of it called category $\mathscr{O}$. These are the representations whose modules have weights bounded above (for some definition of "above").

- We will introduce this category and its basic modules, and show that is has a very important algebraic property: it is a **highest weight category.**
- We will show that this category has a nice presentation; that is, it is equivalent to the modules over a finite dimensional algebra, which we will prescribe combinatorially. This is analogous to the result of Elias and Williamson I mentioned, but *much* easier. This gives us the "combinatorial" column.

- This presentation actually has a very special homological property. It is **Koszul**; this means it only has generators of degree 1, relations of degree 2 (these two properties are just **quadratic**), relations between relations of degree 3, relations between relations between relations of degree 4, etc.
- We'll give a short introduction to Koszul algebras in general and show that the hypertoric category $\mathscr{O}$ is Koszul.

Wednesday, we'll complete the proof of Koszul duality between Higgs and Coulomb categories $\mathscr{O}$ in the abelian case.

- This is going to require some combinatorics; objects in category $\mathscr{O}$ depend on a chamber in a hyperplane arrangement and a choice of a "height" function (remember, we have to know what "bounded above" means). This is what is often called a **linear program.** Our duality piggybacks off an existing result in this setting: the duality theorem in linear programing.
- With all of this background, our "main theorem" is actually pretty easy. Which is good, because we need time to hike.

Thursday, we'll branch out a bit. The first three days will really focus on explaining a specific result: the Koszul duality between hypertoric categories $\mathcal{O}$. However, this result was phrased in purely algebraic terms, and a great deal of the underlying motivation was geometric. We'll try to explain the connection to geometry, as we understand it.

- It's an established principle that quantization can often achieve a similar role to resolution of singularities. The hypertoric varieties we've discussed thus far are affine and singular, but they possess resolutions of singularities, which have their own beautiful geometric structures.
- These smooth varieties have their own theory of deformation quantization (which actually underlies some of the results discussed earlier). This leads to the construction of a quantum structure sheaf on these varieties.

- It's quite interesting to study the sheaves of modules over this quantum structure sheaf; in particular, these are very closely allied with the Fukaya category on the resolution, and they help us avoid some of the pathologies of the algebraic categories.
- This geometric approach also yields a second approach to proving the Koszul duality of the two categories $\mathscr{O}$, using a Ginzburg-Chriss style argument.

On Friday, we'll turn to discussing the non-abelian case. This feels a lot more intimidating, but actually if you really understand the abelian case is not all that much harder.

- We'll give/recall, the definition of the Coulomb and Higgs branches in this more general case.
- We'll review how the approaches we've considered thus far generalize; we'll want to generalize the second argument we've given for duality, since this is more "Higgsy."
- This leads us down some pretty interesting roads, in particular to the KLR algebra, manifested in a pretty surprising way. At the end of that road, we'll find a Koszul duality between category $\mathscr{O}$'s in the general non-abelian case (or at least something close; we can get into the nitty-gritty later).

Mmm, coffee.