Quantum field theory = states of a system are sections of a bundle

We're interested in a particular 3-d theory (so underlying spaces are 3-manifolds) constructed from a group G, ~~and~~ (cpt). and a $\mathbb{C}$-rep V.

Part of data is a principal G-bundle P on 3-mfld M, and a section ϕ of associated bundle $(P \times V)/_G \to M$.

Coulomb branch = Spec of algebra of "local operators" = Spec of Hilbert space of S^2

This is a commutative algebra because of

considered as a cobordism.

expected value of operator = "integral" over "states"

$\Rightarrow$ Hilbert space of surface $= H_*($~~[illegible]~~ translation invariant states on $\mathbb{C} \times \mathbb{R})$.

$\Rightarrow$ local operators $= H_*($ways of transitioning from one state to another, the locally$)$.

This has a product structure via diagram

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transitions transitions

states states states

Since this is local we can think about $\mathbb{R}^3 = \underset{\text{space}}{\mathbb{R}^2} \times \underset{\text{time}}{\mathbb{R}}$! ~~the~~ We look at ~~the~~ the way that we can change the principal bundle and section at the origin from one at $t=-\varepsilon$ and $t=\varepsilon$, which is constant ~~at~~ on $\mathbb{R}^2 \setminus \{(0,0)\}$.

~~We take~~ Consider a formal disk D around $(0,0)$. Let $D^* = D \setminus \{0,0\}$. We have a bundle and section (P_1, ϕ_1) at $t=-\varepsilon$ and one (P_2, ϕ_2) at $t=\varepsilon$. These are isomorphic on D^*. Let us choose a trivialization of this bundle on D^* (which carries from $t=-\varepsilon$ to $t=\varepsilon$).

The only redundancy in this description is the choice of trivialization, which are a torsor over $G((t)) = \mathrm{Maps}(D^* \to G)$. The choice of a principal bundle on D and a trivializ on D^* is parametrized by the affine Grassman $G((t))$ acts transitively on the ~~parametrizations~~ trivializations

~~The obvious trivialization is preserved by $G[[t]]$.~~ The retrivializations of the standard trivialization that extend our D are given by $G[[t]]$. Adding a choice of ϕ is an element of the associated bundle

$$\left(G((t)) \times V[[t]]\right) \big/ G[[t]] = \mathcal{H}_V$$

we need to have ϕ_1 and ϕ_2 agree on D^*; thinking of these as elements of $V((t))$ using the trivialization, is the map $\mathcal{H}_V \to V((t))$

So, the choice of (P_i, ϕ_i) with $\phi_1 = \phi_2$ and the trivialization is given by $\mathcal{H}_V \times_{V((t))} \mathcal{H}_V$; forgetting the trivialization is modding out by $G((t))$.

So we need to consider $H_*^{G((t))}(\mathcal{H}_V \times_{V((t))} \mathcal{H}_V)$

This is the algebra of local operators

The map $\mathcal{H}_V \times_{V[[t]]} \mathcal{H}_V \to \mathcal{H}_V \to Gr$ is $G((t))$-equivariant. Let ${}_v\mathcal{H}_V$ be the fiber over the identity coset ${}_v\mathcal{H}_V = \{(g(t), \phi) \mid g(t)\cdot\phi \in V[[t]]\} / G[[t]]$.

$$\boxed{H_*^{G((t))}(\mathcal{H}_V \times_{V((t))} \mathcal{H}_V) \cong H_*^{G[[t]]}({}_v\mathcal{H}_V).}$$

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What does this homology actually mean? Don't think too hard about it, but here's the rough idea: $\mathcal{H}_V$ is an infinite dimensional vector bundle on an infinite dimensional space.

We want homology classes "finite along base, cofinite along fiber." Actually making this precise is quite annoying: $\mathcal{H}_V$ is an inverse limit of f.d. v.s. and the associated bundles of $V[t]/t^N$ as $N \to \infty$. Every question we consider is insensitive to high enough powers of N. Thus, we can think about the Borel Moore homology in the usual sense of these truncations and then identify them using the pull back isomorphism for N. But don't let this scare you, we just need to use a few simple facts:

1) a finite dimensional subvariety in Gr, ~~and a~~ with a cofinite rank subbundle induces a homo class.

~~2) Pushing forward by a smooth inclusion and then pulling back has the effect of multi~~

3) a clean intersection of homology classes (one that is locally diff. to v.s.) results in the usual intersection times the Euler class of the excess.

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How does this work for a torus?

$$1 \to T_1[[t]] \to T[[t]] \to T \to 1$$

The group $T_1[[t]]$ is contractible.

$$1 \to T[[t]] \to T((t)) \to X_*(T) \to 1$$

$$D((t)) = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} \qquad a_i(t) = b^{(i)}_{k_i} t^{k_i} + b^{(i)}_{k_i+1} t^{k_i+1} + \dots \qquad b^{(i)}_{k_i} \neq 0$$

$$D[[t]] = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} \qquad k_i = 0$$

$$X_*(T) \quad \begin{bmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_n \end{bmatrix} \longleftarrow\!\shortmid \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$$

$$D_1[[t]] \quad \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} \qquad k_i = 0,\ b^{(i)}_0 = 1$$

Thus $T((t))/T[[t]] \cong X_*(T)$
$[t^{\nu}]$ for $\gamma \in X_*(T)$

Let $V_\mathcal{O} = V[[t]]$. We have

$$\mathcal{X}_V = \bigsqcup_{\lambda \in X_*(T)} \{t^\lambda\} \times t^\lambda \cdot V_\mathcal{O}$$

$${}_V\mathcal{X}_V = \bigsqcup_{\lambda \in X_*(T)} \{t^\lambda\} \times \left(V_\mathcal{O} \cap t^\lambda \cdot V_\mathcal{O}\right)$$

As an abelian group, there's nothing interesting about the homology. Let $r_\lambda = \left[\{t^\lambda\} \times (V_\mathcal{O} \cap t^\lambda V_\mathcal{O})\right]$

$$H_*^{G_\mathcal{O}}({}_V\mathcal{X}_V) = \bigoplus H^*(BT) \cdot r_\lambda$$

The only interesting thing is the multiplication. Left and right multiplication by $H^*(BT)$ corresponds to the tautological bundles on the left and right copies of $\mathcal{X}_V$. Transfering to ${}_V\mathcal{X}_V$, right multiplication corresponds to the Euler classes of tautological bundles, and left to the trivial bundle with the character giving the equivariant structure.

In this case, the tautological bundles are trivial, so the left and right actions are the same.

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So, the only interesting product is $r_\lambda r_\nu$. We pullback their classes to $\mathcal{X}_V \times \mathcal{X}_V \times \mathcal{X}_V$, and then intersect them

$$p_{12}^* r_\lambda = \left[\begin{matrix}(x_1+\lambda,\ x_1,\ x_2)\\ (v_1(t),\ v_1(t),\ v_2(t))\end{matrix}\right] \qquad p_{23}^* r_\nu \left[\begin{matrix}(\xi_1,\ \xi_2+\nu,\ \xi_2 \ldots\\ (w_1(t),\ w_2(t),\ w_2(t) \ldots\end{matrix}\right.$$

The naive intersection is

$$\left[\begin{matrix}(x+\lambda+\nu,\ x+\nu,\ x)\\ (v(t),\ v(t),\ v(t))\end{matrix}\right] \quad \text{w/} \quad v(t) \in t^{x+\lambda+\nu} V_{\mathcal{O}} \cap t^{x+\nu} V_{\mathcal{O}} \cap t^{x} V_{\mathcal{O}}$$

However, we have to account for the excess bundle, given by

$$t^{x+\lambda+\nu} V_{\mathcal{O}} + t^{x+\nu} V_{\mathcal{O}} + t^{x} V_{\mathcal{O}} \Big/ \left(t^{x+\lambda+\nu} V_{\mathcal{O}} \cap t^{x+\nu} V_{\mathcal{O}}\right) + \left(t^{x+\nu} \ldots\right) \cap t^{x} V_{\mathcal{O}} \ldots$$

If v is of weight μ, then $t^n v \in t^x V_{\mathcal{O}}$ if $n \geq \langle x, \mu \rangle$. Thus $t^n v$ contributes to the excess if $n \geq \langle x+\nu, \mu\rangle$ and $n < \langle x, \mu\rangle$ and $n < \langle x+\lambda+\nu, \mu\rangle$, and we get a factor of μ for each such n.

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Note, this requires $\langle \nu, \mu \rangle < 0$ and $\langle \lambda, \mu \rangle > 0$.

In addition, we must push forward by the map forgetting the middle element. This sends us to the class of

$$\left(\begin{pmatrix} x+\nu+\lambda, & x \\ v(t) & , v(t) \end{pmatrix} \right) . \quad v(t) \in t^{x+\nu+\lambda} V_{\mathcal{O}} \cap t^{x+\nu} V_{\mathcal{O}} \cap t^{x} V_{\mathcal{O}}$$

This is almost $r_{\nu+\lambda}$, but we have the normal bundle

$$t^{x+\nu+\lambda} V_{\mathcal{O}} \cap t^{x} V_{\mathcal{O}} \Big/ t^{x+\nu+\lambda} V_{\mathcal{O}} \cap t^{x+\nu} V_{\mathcal{O}} \cap t^{x} V_{\mathcal{O}}$$

~~Now, we need~~ for $t^n v$ to contribute where, we must have $\langle x+\nu, \mu \rangle \geq n$ and $\langle x, \mu \rangle \leq n$ and $\langle x+\lambda+\nu \rangle \leq n$. This requires $\langle \nu, \mu \rangle > 0$ and $\langle \lambda, \mu \rangle < 0$.

Thus, we have $r_\lambda r_\nu = r_{\lambda+\nu} \cdot \prod e_i^{d(\langle \nu, e_i \rangle, \langle \lambda, e_i \rangle)}$

where $d(k,l) = \begin{cases} 0 & k, l \text{ have same sign} \\ \min(|k|, |l|) & k, l \text{ have different signs} \end{cases}$

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This is the multiplication rule in the classical limit of the hypertoric enveloping algebra. In order to get the algebra itself, we ~~???~~ go back to the physics. There, we quantize using an Ω-background corresponding to rotation of $\mathbb{R}^2$ in $\mathbb{R}^2 \times \mathbb{R}^1$.

On the affine Grassmannian, this becomes the loop rotation action of $\mathbb{C}^*$.

We have to consider

$$H_*^{\mathbb{F}(K) \rtimes \mathbb{C}^*}(\mathcal{X}_V \times_{V_\mathcal{O}} \mathcal{X}_V) \cong H_*^{\mathbb{F}(\mathcal{O}) \rtimes \mathbb{C}^*}({}_V\mathcal{X}_V)$$

and we have to remember the loop action on everything. For example

$\mu \cdot r_\lambda$ = tautological bundle for $\mathbb{C}_\mu$ on $(\chi+\lambda, \chi)$ pulled back from the second factor.
Loop $\mathbb{C}^*$ acts w/ weight $\langle \chi+\lambda, \mu \rangle$

$r_\lambda \cdot \mu$ = ______________________________
______________________ first factor
Loop $\mathbb{C}^*$ acts w/ weight $\langle \chi, \mu \rangle$

Letting h be equivariant parameter for loop $\mathbb{C}^*$

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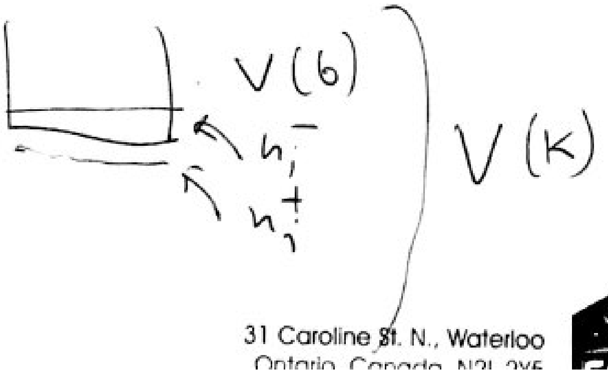
~~Thus, Theorem The convolution algebra~~ ~~$H_*^{G(6)}(\mathcal{Z}\mathcal{H}_v)$~~ ~~is the classical~~

This result can be extended in several ways. We can consider equivariance for different groups. The full diagonal matrices H act on V, so we can take $H_*^{H\times G}(v\mathcal{H}_v)$ as well. Secondly, the disk has a $\mathbb{C}^*$-action by rotation ($\mathbb{R}^2 \cong \mathbb{C}$), which induces an action on $\mathcal{H}_v$, $v\mathcal{H}_v$, etc. This does not commute with $G(K)$; instead we get an action of the semi-direct product $G(K)\rtimes \mathbb{C}^*$ induced by $s\cdot t^a = s^a t^a$.

I'm going to identify ~~$H^* B(H \times \mathbb{C}^*)$~~ $H^*(B(H(\mathcal{O})\rtimes\mathbb{C}^*)) = H^*(B(H\times\mathbb{C}^*))$ with $D_0[h]$. h_i^- corresponds to the weight of the ith coordinate in $\mathbb{C}^n$ w/ $\mathbb{C}^*$ acting trivially and h_i^+ to the same coordinate w/ $\mathbb{C}^*$ acting w/ weight -1.



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If we turn on the loop $\mathbb{C}^*$, this is what physicists call an Ω-background and we have to careful about loop weights, when computing Euler classes. To get right mult, take $\chi = 0$; to get left, $\chi = -\lambda - \nu$.

Consider $\chi = 0$. Then we get

$$r_\lambda r_\nu = r_\lambda r_\nu \cdot \mathrm{Eu}\left(t^{\lambda+\nu} V_O + t^{\nu} V_O + V_O / (t^{\lambda+\nu} V_O \cap t^{\nu} V_O) + (t^{\nu} V_O \cap V_O\right.$$

$$\cdot \mathrm{Eu}\left(t^{\lambda+\nu} V_O \cap V_O / t^{\lambda+\nu} V_O \cap t^{\nu} V_O \cap V_O\right)$$

~~If $\langle \lambda+\nu, \mu \rangle = 0$, then $t^{\lambda+\nu} V_O \cap V_O = V_O$, and a vector of weight μ contributes~~

If $\langle \lambda, \mu \rangle$ and $\langle \nu, \mu \rangle$ have same sign, or one is zero, no contribution. If they have opposite then

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	$\langle \lambda+\nu, \mu\rangle \leq 0$	$\langle \lambda+\nu, \mu\rangle \geq 0$
$\langle \nu, \mu\rangle \geq 0$	$v, \text{---}, t^{\langle \nu,\mu\rangle -1} v$	$t^{\langle \lambda+\nu,\mu\rangle} v, \text{---}, t^{\langle \nu,\mu\rangle-1} v$
$\langle \nu, \mu\rangle < 0$	$t^{\langle \nu,\mu\rangle} v, \text{---}, t^{\langle\lambda+\nu,\mu\rangle -1} v$	$t^{\langle \nu,\mu\rangle}, \text{---}, t^{-1} v$

$[h_i]^{-\langle \nu,\mu\rangle}$	$[h_i]^{+\langle \lambda,\mu\rangle}$ shifted by $\lambda+\nu$
$[h_i]^{+\langle \lambda,\mu\rangle}$ shifted by $\lambda+\nu$	$[h_i]^{-\langle \nu,\mu\rangle}$

Thus, if we want to be cleaner, we can move the terms where $|\langle \lambda,\mu\rangle| < |\langle \nu,\mu\rangle|$ to the other side, and get Nick's formula

$$r_\lambda r_\nu = \prod_{\substack{|\langle \lambda,\varphi_i\rangle| < |\langle \nu,\varphi_i\rangle| \\ \langle \lambda,\varphi_i\rangle\langle \nu,\varphi_i\rangle < 0}} [h_i]^{+\langle \lambda,\mu\rangle} \, r_{\lambda+\nu} \, [h_i]^{-\langle \nu,\mu\rangle}$$

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Thus, we see

Thm $H_*^{H(G) \times G^a}(\nu \not\times \nu)$ with $h=1$ is the hypertoric enveloping algebra

Note an important subtlety here: my r_λ ~~coc~~ corresponds to a cocharacter of T; Nick's m^λ corresponded to a character. Thus, my T is $T^\vee$ in Nick's notation.

Nick $1 \to K \to (\mathbb{C}^*)^n \to T \to 1$

Me $1 \to T^\vee \to (\mathbb{C}^*)^n \to K^\vee \to 1$

Switching between $K \subset (\mathbb{C}^*)^n$ and $((\mathbb{C}^*)^n/K)^\vee$ is called Gale duality. It'll be important for us moving forward

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