QUANTIZATION EXERCISES

(1) The special linear group SL_2 acts on $\mathbb{P}^1$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x_0, x_1] = [ax_0 + bx_1, cx_0 + dx_1].$$

(a) By differentiating this action action construct Lie algebra anti-homomorphism

$$\alpha : \mathfrak{sl}_2 \to \Gamma(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1})$$

This is an anti-homormophism because the standard Lie algebra structure on $\Gamma(X, \mathcal{T}_X)$ is the opposite of the one you get from the identification

$$\mathrm{Lie}(Aut(X)) \cong \Gamma(X, \mathcal{T}_X).$$

(b) Show that $-\alpha$ induces a surjection $\eta : U(\mathfrak{sl}_2) \cong \Gamma(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1})$.

(c) Show that the Casimir $\frac{1}{2}h^2 + ef + fe$ goes to a scalar under η.

(d) The classical limit of η gives a map $T^*\mathbb{P}^1 \to (\mathfrak{sl}_2)^*$. Show that the image of this map is the nilpotent cone. You will need to use the Killing form to identify $\mathfrak{sl}_2$ with it's dual.

(2) A similar technique can be used to construct a map from $U(\mathfrak{sl}_2)$ to the hypertoric enveloping algebra U for $G = \mathbb{C}^*_\Delta \to (\mathbb{C}^*)^2$. Recall that the central quotients of U are parameterized by points $\lambda \in \mathfrak{g}^*$.

(a) Find a formula that gives the value of the Casimir in each U_λ.

(b) Which value of λ gives the unique self opposite quantization? Draw the corresponding quantum arrangement.