WARTHOG 2017

The nonabelian Higgs and Coulomb branches

Exercises

Consider the Nakajima quiver variety associated to the Dynkin diagram of $\mathfrak{sl}_2$:

$$\begin{tikzcd} W = \mathbb{C}^n \arrow[d, bend right, "\eta"'] \\ V = \mathbb{C}^k \arrow[u, bend right, "\varepsilon"'] \end{tikzcd}$$

So $\eta \in \mathrm{Hom}(W, V)$, $\varepsilon \in \mathrm{Hom}(V, W)$, with the natural action of $GL(V)$.

For the usual choice of GIT parameter ξ, the corresponding Nakajima quiver variety is

$$\mathcal{M} := \mathcal{M}_{H,\xi} = \{[\eta, \varepsilon] : \eta\varepsilon = 0, \quad \varepsilon \text{ is injective}\}$$

where $[\eta, \varepsilon]$ denotes the equivalence class under the relation

$$[\eta, \varepsilon] = [\eta g^{-1}, g\varepsilon], \qquad \forall g \in GL(V)$$

(a) Show that $\mathcal{M} \cong T^*\mathsf{Gr}(k, n)$.

(*Hint:* There is a map $\mathcal{M} \to \mathsf{Gr}(k, n)$ defined by sending $[\eta, \varepsilon]$ to the subspace $\mathrm{image}(\varepsilon) \subset \mathbb{C}^n$.)

(b) Define a map

$$\mathcal{M} \to \mathrm{End}(W) = M_{n\times n}(\mathbb{C}), \qquad [\eta, \varepsilon] \mapsto X := \varepsilon\eta$$

Show that any X in the image of this map is nilpotent. What equations does it satisfy? What are the possible Jordan forms for X?

(c) There is a residual action of $GL(W)$ on $\mathcal{M}$, which we can transfer to $T^*Gr(k, n)$. Consider an integral cocharacter $\xi : \mathbb{C}^\times \to T$, where $T \subset GL(W)$ is the maximal torus of diagonal matrices. What does the fixed point locus look like in $T^*Gr(k, n)$?

How many irreducible components are there?