WARTHOG 2017

Koszul Duality

Exercises

1. (a) Compute (carefully!) the quadratic qual of the quiver algebra for category $\mathcal{O}$. (There are two vertices, and two arrows named f and g, with the relation $fg = 0$.)
   (b) What is the quadratic dual of the analogous quiver algebra when you kill both fg and gf? Is it Koszul?

2. Let Λ be the exterior algebra in two variables. Compute the linear projective resolution of the ground field.

3. Prove that the differentials in a linear projective resolution of $\mathbb{k}$ (as a left A-module), and the augmentation map to $\mathbb{k}$, are all $\mathbb{k}$-bimodule maps.

4. This exercise covers the details of the proof that the category $\mathcal{LP}(A)$ is equivalent to the category of graded $A^!$-modules.

   (a) Let $\{\alpha_i\}$ and $\{\beta_i\}$ be dual bases of V and V^* respectively. Prove that the element $\sum_i \alpha_i \otimes \beta_i \in V \otimes V^*$ is independent of the choice of dual bases. (Hint: Consider the map $V \otimes V^* \otimes V \to V$ which sends $v \otimes f \otimes w \mapsto vf(w)$. What does this do to $\sum_i \alpha_i \otimes \beta_i \otimes v$, for $v \in V$?)

   (b) Suppose X is a graded $A^!$-module. Let d be the A-module map $A \otimes X_i \to A \otimes X_{i+1}$ of graded degree $+1$ sending
   $$d(a \otimes x) = \sum_i a\alpha_i \otimes \beta_i x.$$
   Show that $d^2 = 0$, as a degree two map from $A \otimes X_i \to A \otimes X_{i+2}$. Thus one has a complex $P^\bullet$ in $\mathcal{LP}(A)$, where $P^i = A\langle i \rangle \otimes X_i$.

   (c) Conversely, let $P^\bullet$ be an object in $\mathcal{LP}(A)$, with $P^i = A\langle i \rangle \otimes X_i$. We aim to give $X = \oplus X_i$ the structure of an $A^!$-module. For $\beta \in V^*$ and $x \in X_i$ define $\beta(x) = \sum \beta(a_i) y_i \in X_{i+1}$, where $d(1 \otimes x) = \sum a_i \otimes y_i$. Confirm that the quadratic relations in $R^!$ are satisfied by the operators β.

   (d) Confirm that these operations, sending $A^!$-modules to linear complexes and back, are inverse to each other.

   (e) Confirm that the grading shift $\langle 1 \rangle$ on an $A^!$-module is intertwined with the twist $\langle -1 \rangle[1]$ in $\mathcal{LP}(A)$.

5. Let S be the polynomial algebra in two variables, with quadratic dual Λ. Let X be the graded S-module $S/(x^3, x^2y, y^2)$. What is the corresponding linear complex of projective Λ-modules?

6. Verify that ${}^*(A^!_n)$ is equal to the intersection of the subspaces $V \otimes \cdots \otimes V \otimes R \otimes V \otimes \cdots \otimes V$ inside $V^{\otimes n}$.

7. Prove that $A^! = \mathrm{Ext}^*_A(\mathbb{k}, \mathbb{k})^{\mathrm{op}}$.