Lecture 6

Regular sequences in the ultraproduct ring

Recall that R_∞ is the infinite variable polynomial ring $k[x_1, x_2, \ldots]$, and $\mathbf{S}$ is the graded ultrapower of R_∞. Finitely generated ideals of $\mathbf{S}$ can be seen as families of ideals in the R_∞'s. Precisely, a family of homogeneous ideals $\{I_i\}$ of R_∞ is *uniformly finitely generated* if there is an n and d such that each I_i is generated by $\leq n$ elements of degrees $\leq d$. In this case, the graded ultraproduct of the I_i's is a finitely generated ideal of $\mathbf{S}$. If $\{J_i\}$ is another such sequence with the same ultraproduct then $I_i = J_i$ for all i in some set in the ultrafilter. Every finitely generated ideal of $\mathbf{S}$ arises in this manner. (This stuff is a little technical, and probably shouldn't be proved in lecture.)

Main theorem for this lecture [ESS2, Theorem 4.8]:

Theorem 6.1. *If I is a finitely generated ideal of $\mathbf{S}$ then $\operatorname{codim}_{\mathbf{S}}(I) = \operatorname{codim}_{R_\infty}(I_i)$ for all i in a set in the ultrafilter.*

Proof. We argue as follows:

- Let $c = \operatorname{codim}_{\mathbf{S}}(I)$, which is finite (Corollary 5.3). We proceed by induction on c. The $c = 0$ case is trivial. Now assume the result for $c-1$ and prove it for $c > 0$.
- Let $f = (f_i)_{i \in I}$ be an element of I. We can find an automorphism γ_i of R_i such that $\gamma_i(f_i)$ is monic in x_1. The γ_i induce an automorphism γ of $\mathbf{S}$. Replacing I with $\gamma(I)$, we can assume that f_i is monic in x_1 for all i.
- Let $R_i = k_i[x_2, x_3, \ldots]$ and let $\mathbf{S}'$ be their graded ultraproduct; we have $\mathbf{S} = \mathbf{S}'[x_1]$. Let $I' = I \cap \mathbf{S}$, which is finitely generated.
- Claim: $I'_i = R'_I \cap I_i$ for all i in some set in the ultrafilter. See [ESS2, Prop 4.5].
- Result now follows. By generalities on codimension we have $\operatorname{codim}_{\mathbf{S}'}(I') = \operatorname{codim}_{\mathbf{S}}(I) - 1$ and $\operatorname{codim}_{R'_i}(I'_i) = \operatorname{codim}_{R_i}(I_i) - 1$. By induction, we have $\operatorname{codim}_{\mathbf{S}'}(I') = \operatorname{codim}_{R'_i}(I'_i)$, and so the result follows. ☐

Remark 6.2. It is worth emphasizing that all the dimension theory used above only works since we know that $\mathbf{S}$ is a polynomial ring!

In fact, we are most interested in the following corollary of the theorem [ESS2, Corollary 4.10]:

Corollary 6.3. *Let $f_1, \ldots, f_r$ be homogeneous elements of $\mathbf{S}$. Then $f_1, \ldots, f_r$ form a regular sequence in $\mathbf{S}$ if and only if $f_{1,i}, \ldots, f_{r,i}$ form a regular sequence in R_∞ for all i in some set in the ultrafilter.*

Proof. This follows from the theorem and the characterization of regular sequences via codimension in Proposition 5.6. ☐

Exercises

Exercise 6.1. Let $f_{1,\bullet}, \ldots, f_{r,\bullet}$ and $g_{1,\bullet}, \ldots, g_{s,\bullet}$ be sequences of homogeneous elements of R_∞ indexed by I, and let $f_1, \ldots, f_r$ and $g_1, \ldots, g_s$ be the corresponding elements of $\mathbf{S}$. Define

$$\mathfrak{a}_i = (f_{1,i}, \ldots, f_{r,i}) \qquad \mathfrak{b}_i = (g_{1,i}, \ldots, g_{s,i})$$
$$\mathfrak{a} = (f_1, \ldots, f_r) \qquad \mathfrak{b} = (g_1, \ldots, g_s).$$

Show that $\mathfrak{a} = \mathfrak{b}$ if and only if $\mathfrak{a}_i = \mathfrak{b}_i$ for all i belonging to some set in the ultrafilter.

Exercise 6.2. Let $\mathfrak{a}$ be a finitely generated ideal of $\mathbf{S}$ and let $\mathfrak{a}_\bullet$ be the corresponding sequence of ideals in R_∞. Show that $\mathfrak{a}$ is radical if and only if $\mathfrak{a}_i$ is radical for all i in some set in the ultrafilter. To do this, make use of the following result:

(∗) Given degrees $d_1, \ldots, d_r$ there exists a positive integer $M = M(d_1, \ldots, d_r)$ with the following property: if $\mathfrak{a}$ is an ideal in a polynomial ring generated by r homogeneous elements of degrees $d_1, \ldots, d_r$ then $\mathrm{rad}(\mathfrak{a})^M \subset \mathfrak{a}$.

This statement is a consequence of the *effective Nullstellensatz*.

Exercise 6.3. Prove a version of Theorem 6.1 for the inverse limit ring $\mathbf{R}$.

Notes

[ESS2, §4].