Lecture 6: main exercises

Exercise 6.1. Let $f_{1,\bullet}, \dots, f_{r,\bullet}$ and $g_{1,\bullet}, \dots, g_{s,\bullet}$ be sequences of homogeneous elements of R_∞ indexed by I, and let $f_1, \dots, f_r$ and $g_1, \dots, g_s$ be the corresponding elements of $\mathbf{S}$. Define

$$\mathfrak{a}_i = (f_{1,i}, \dots, f_{r,i}) \qquad \mathfrak{b}_i = (g_{1,i}, \dots, g_{s,i})$$
$$\mathfrak{a} = (f_1, \dots, f_r) \qquad \mathfrak{b} = (g_1, \dots, g_s).$$

Show that $\mathfrak{a} = \mathfrak{b}$ if and only if $\mathfrak{a}_i = \mathfrak{b}_i$ for all i belonging to some set in the ultrafilter.

Exercise 6.2. Let $\mathfrak{a}$ be a finitely generated ideal of $\mathbf{S}$ and let $\mathfrak{a}_\bullet$ be the corresponding sequence of ideals in R_∞. Show that $\mathfrak{a}$ is radical if and only if $\mathfrak{a}_i$ is radical for all i in some set in the ultrafilter. To do this, make use of the following result:

(∗) Given degrees $d_1, \dots, d_r$ there exists a positive integer $M = M(d_1, \dots, d_r)$ with the following property: if $\mathfrak{a}$ is an ideal in a polynomial ring generated by r homogeneous elements of degrees $d_1, \dots, d_r$ then $\mathrm{rad}(\mathfrak{a})^M \subset \mathfrak{a}$.

This statement is a consequence of the *effective Nullstellensatz*.

Exercise 6.3. Prove a version of the main theorem (about codimension in $\mathbf{S}$) for the inverse limit ring $\mathbf{R}$.