Lecture 5: main exercises

Exercise 5.1. Let $I = (xz, yz) \subset k[x, y, z]$. Determine the minimal prime ideals containing I and compute their codimensions.

Exercise 5.2. Let I be a finitely generated ideal in an infinite polynomial ring R. Show that I has infinite dimension. [This is intuitively obvious, but to actually prove it you have to exhibit long chains of prime ideals containing I.]

Exercise 5.3. Find three non-zero polynomials f_1, f_2, f_3 in $k[x, y, z]$ that are pairwise coprime but that do not form a regular sequence.

Lecture 5: additional exercises

Exercise 5.4. Let $\mathfrak{p}$ be a prime of finite codimension c in a polynomial ring (over a field). Show that $\mathfrak{p}$ is finitely generated. [Hint: to do the $c = 1$ case, contract down to a finite variable polynomial ring containing a non-zero element of $\mathfrak{p}$ and then extend back. The general case can be proved by induction on c, using a similar idea.]

Exercise 5.5. Prove the key proposition from lecture: extending ideals from a polynomial ring to a large polynomial ring does not change codimension. First treat the case where I is prime. Then do the general case; you may need to make use of Exercise 5.4 in the general case.