Lecture 1: main exercises

Exercise 1.1. Determine the strength of $x_1^2 + \cdots + x_n^2$ over the complex numbers.

Exercise 1.2. Let R be any graded k-algebra and let R_+ be the ideal of positive degree elements in R.

(a) Show that a homogeneous element has finite strength if and only if it belongs to R_+^2. In particular, the finite strength elements form an ideal.
(b) Show that a collection of homogeneous elements has infinite collective strength if and only if it maps to a k-linearly independent set in R_+/R_+^2.
(c) Show that the homogeneous elements of R of infinite strength generate R as a ring. Even better, show that is a family of elements $\{f_i\}_{i\in I}$ of infinite collective strength that generates R.

Lecture 1: additional exercises

Exercise 1.3. Let $f \in k[x_1, \dots, x_n]$ be homogeneous of positive degree d with k a field of characteristic 0, and let J be the ideal generated by the partial derivatives of f.

(a) Prove *Euler's formula*: $f = \frac{1}{d} \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}$.
(b) If f has strength $\leq s$ show that J is contained in an ideal generated by $\leq 2s$ elements.
(c) Conversely, if J is contained in an ideal generated by $\leq s$ elements show that f has strength $\leq s$.

Exercise 1.4. Show that $k = R_n/(x_1, \dots, x_n)$ has projective dimension n as an R_n-module.

Exercise 1.5. Let $V_1, \dots, V_n$ be vector spaces. Let $x \in V_1 \otimes \cdots \otimes V_n$. Recall that x is a *pure tensor* if $x = v_1 \otimes \cdots \otimes v_n$ for some $(v_1, \dots, v_n) \in V_1 \times \cdots \times V_n$. We say that x has *tensor rank* $\leq r$ if x is a sum of $\leq r$ pure tensors.

(a) Show that the tensor rank of x is at most $\prod_{i=1}^n \dim V_i$, assuming each V_i is finite dimensional.
(b) Let V be finite dimensional. Recall there is a natural isomorphism $\mathrm{End}(V) = V \otimes V^*$. Show that the tensor rank of an element of $V \otimes V^*$ coincides with its usual rank as an endomorphism of V.
(c) Again, let V be finite dimensional, and suppose k has characteristic $\neq 2$. We can identify the space $\mathrm{Sym}^2(V)$ of degree 2 polynomials in V with a subspace of $V^{\otimes 2}$. How do strength and tensor rank compare?

Exercise 1.6. Show that the polynomial $\sum_{i=1}^n x_i y_i z_i$ has strength n. (This is not so easy; see [DES, §4] for a proof.)