Exercises

Exercise 15.1. Let I be a finitely generated ideal of polynomial ring (such as $\mathbf{R}$ or $K \otimes_A \mathbf{R}$). Show that $\mathrm{pdim}_{\mathbf{R}}(I)$ is finite. [This is elementary: you don't need to use anything like Stillman's conjecture.]

Let A be an integral k-algebra with fraction field K.

Exercise 15.2. How are the rings $\mathbf{R}_K$ and $K \otimes_A \mathbf{R}_A$ related? [Is there a homomorphism? Is it injective/surjective/isomorphism?]

Exercise 15.3. Show that $K \otimes_A \mathbf{R}_A$ is a polynomial K-algebra. [Hint: use the derivational criterion.]

Additional exercises

Exercise 15.4. Regard the polynomial algebra $A[x_1, \ldots, x_n]$ as a graded ring where A has degree 0 and each x_i has degree 1. Suppose that M is a finitely presented graded A-module. Show that there is an open dense subset U of $\mathrm{Spec}(A)$ such that the Betti table of M_x is constant for $x \in U$.

[Hint: consider the resolution of $K \otimes_A M$ over $K[x_1, \ldots, x_n]$. The differentials are matrices with entires in K. They therefore belong to $A[1/f]$ for some non-zero $f \in A$. In this way, one can spread out the complex to $A[1/f]$. Further argument is needed to show that one can choose f so that the complex is exact, and that the projective dimension doesn't go down.]

Exercise 15.5. Deduce the classical form of Stillman's conjecture from the version of Stillman's conjecture for $\mathbf{R}$.