Exercises

Exercise 13.1. Let X be the closed $\mathbf{GL}$-subvariety of $\mathbf{A}^{[(1),(1)]}$ consisting of linearly dependent pairs. Explicitly verify the conclusion of the shift theorem in this case.

Exercise 13.2. Let X be the rank $\le s$ locus in $\mathbf{A}^{(2)}$. Explicitly verify the conclusion of the shift theorem in this case.

Exercise 13.3. Show that $\mathrm{Sh}_n(\mathbf{S}_\lambda)$ has the form $\mathbf{S}_\lambda \oplus \cdots$, where the remaining terms are Schur functors of smaller degree.

Additional exercises

Exercise 13.4. Let X be a $\mathbf{GL}$-variety. Show that the invariant function field $k(X)^{\mathbf{GL}}$ is a finitely generated extension of k. (See Exercise 12.4 for the definition of $k(X)^{\mathbf{GL}}$.) [Hint: use the shift theorem (Theorem 13.2).]

Exercise 13.5. Explicitly compute $\mathrm{Sh}_n(\mathbf{S}_\lambda)$ in terms of Littlewood–Richardson coefficients (if you know what these are).

Exercise 13.6. Let X be an affine $\mathbf{GL}$-variety.

(a) Show that there is a natural surjective map of $\mathbf{GL}$-varieties $\mathrm{Sh}_n(X) \to X$. [Hint: this is induced by the canonical inclusion $\mathbf{V} \to \mathrm{Sh}_n(\mathbf{V})$.]
(b) Show that there is a dominant morphism $B \times \mathbf{A}^{\underline{\lambda}} \to X$ for some finite dimensional variety B and some tuple $\underline{\lambda}$. [This says that X is "unirational up to a finite dimensional error."]