Exercises

Exercise 12.1. Describe all closed **GL**-subvarieties of $\mathbf{A}^{(1)} \times \mathbf{A}^{(1)}$.

Exercise 12.2. Show that the rank $\leq s$ loci account for all the non-empty proper closed **GL**-subvarieties of $\mathbf{A}^{(2)}$.

Example 12.7. Let $\underline{\lambda} = [(d_1), \ldots, (d_r)]$. Show that any map of **GL**-varieties $\varphi \colon \mathbf{A}^{\underline{\lambda}} \to \mathbf{A}^{(e)}$ has the form

$$\varphi(f_1, \ldots, f_r) = \Phi(f_1, \ldots, f_r)$$

where $\Phi \in k[T_1, \ldots, T_r]$ is polynomial. Moreover, show that Φ is homogeneous of degree e if T_i is given degree d_i. (A map of **GL**-varieties is simply a **GL**-equivariant map of schemes over k.)

Additional exercises

Exercise 12.3. A point in a $\mathbf{GL}$-variety is $\mathbf{GL}$-*generic* if it has dense $\mathbf{GL}$-orbit.

(a) Show that any element of $\mathbf{A}^{(2)}$ of infinite strength is $\mathbf{GL}$-generic. In fact, this is true in $\mathbf{A}^{(d)}$ as well, but the proof is much harder.

(b) Let $f = \sum_{i \geq 0} x_{3i+1}x_{3i+2}x_{3i+3}$, regarded as a point of $\mathbf{A}^{(3)}$. Show that f is $\mathbf{GL}$-generic. (Hint: show that any cubic in n variables can be obtained as a limit of points in the $\mathbf{GL}$-orbit of f.)

(c) On the other hand, show that there is no point in $\mathrm{Sym}^3(k^n)$ with dense $\mathbf{GL}_n$-orbit. Thus the above result is somewhat surprising.

(d) Show that if $\underline{\lambda}$ is any tuple of non-empty partitions the space $\mathbf{A}^{\underline{\lambda}}$ admits a $\mathbf{GL}$-generic point.

Exercise 12.4. Let $X = \mathrm{Spec}(R)$ be an irreducible $\mathbf{GL}$-variety. The *invariant function field* of X is the field $k(X)^{\mathbf{GL}}$ of $\mathbf{GL}$-invariant elements in the function field $k(X) = \mathrm{Frac}(R)$. Compute this when X is the closed subvariety of $\mathbf{A}^{(1)} \times \mathbf{A}^{(1)}$ consisting of pairs (x, y) that are linearly dependent.