Exercises

Exercise 11.1. Let V be a finite length polynomial representation of $\mathbf{GL}$. Show that the infinite symmetric group $\mathfrak{S}$ acts with finitely many orbits on the set of weights appearing in V.

Exercise 11.2. Let V be a polynomial representation of $\mathbf{GL}$ and let V_{1^k} denote its 1^k weight space under $\mathbf{T}$. (1^k means the weight $(1, \ldots, 1, 0, 0, \ldots)$, where there are k 1's.)

(a) Show that V_{1^k} is naturally a representation of the symmetric group $\mathfrak{S}_k$.
(b) When $V = \mathbf{S}_\lambda(\mathbf{V})$ show that V_{1^k} is the Specht module $\mathbf{Sp}_\lambda$.

Additional exercises

Exercise 11.3. Let $\mathrm{Rep}(\mathfrak{S}_*)$ be the category whose objects are sequences $(M_n)_{n\geq 0}$ where M_n is a complex representation of the symmetric group $\mathfrak{S}_n$.

(a) Show that there is a unique (up to isomorphism) equivalence of categories

$$\Phi\colon \mathrm{Rep}(\mathfrak{S}_*) \to \mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$$

satisfying $\Phi(\mathbf{Sp}_\lambda) = \mathbf{S}_\lambda(\mathbf{V})$. (Here we regard $\mathbf{Sp}_\lambda$ as the sequence that is only non-zero in index $|\lambda|$.)

(b) Via Φ, we can transport the tensor product on $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ to $\mathrm{Rep}(\mathfrak{S}_*)$. What do we get?