4 - 1 : Deligne-Lusztig characters

(0): Notation

G — (a conn red gp $/\mathbb{F}_q$) $\underset{\mathbb{F}_q}{\otimes} \overline{\mathbb{F}_q}$

F — the associated geom Frob on G
($G^F := G(\overline{\mathbb{F}_q})^F$)

$T \subseteq G$ — a max. torus s.t. $FT \subseteq T$.

$B = U \rtimes T \subseteq G$ — Levi decomp of a Borel B
s.t. $FB \subseteq B$.

$W := N_G(T)/T$ (fix a choice of representatives)

$$
\begin{array}{ccc}
W & & \\
\rotatebox{90}{$\in$} & & \\
w & \rightsquigarrow & \tilde{X}(w) := \{gU \in G/U \mid g^{-1}F(g) \in UwU\} \\
& & \big\downarrow /T^{wF} \quad (:= \{t \in T \mid F(t) = t^w\}) \\
& & X(w) := \{gB \in G/B \mid g^{-1}F(g) \in BwB\}
\end{array}
$$

(a finite cover by T^{wF}
(finite étale surjection))

$\theta \in \mathrm{Irr}_{\overline{\mathbb{Q}_\ell}} T^{wF}$ — an arbitrary irr char

V_θ — the vec sp $/\overline{\mathbb{Q}_\ell}$ corresponding to θ.

(1): Deligne–Lusztig characters

Recall: $G^F \circlearrowright \tilde{X}(w) \circlearrowleft T^{wF}$

$$\rightsquigarrow \quad G^F \circlearrowright H_c^i(\tilde{X}(w), \overline{\mathbb{Q}}_\ell) \circlearrowleft T^{wF}$$

I.e. $H_c^i(\tilde{X}(w), \overline{\mathbb{Q}}_\ell)$ are (G^F, T^{wF}) - bimodules

<u>Definition</u>: The virtual rep

$$R_w^G(\theta) := \sum_{i\in\mathbb{Z}} (-1)^i \left(H_c^i(\tilde{X}(w), \overline{\mathbb{Q}}_\ell) \otimes_{\overline{\mathbb{Q}}_\ell T^{wF}} V_\theta \right)$$

is called the <u>Deligne-Lusztig rep</u> associated to θ.

<u>Remark</u>: As the notation suggested, $R_w^G(\theta)$ is independent of the choice of B.

<u>Example</u>: Let $\theta = 1$ be the trivial irrep, then

$$H_c^i(\tilde{X}(w), \overline{\mathbb{Q}}_\ell) \otimes_{\overline{\mathbb{Q}}_\ell T^{wF}} \overline{\mathbb{Q}}_\ell = H_c^i(X(w), \overline{\mathbb{Q}}_\ell)$$

$$\Rightarrow \underline{\underline{R_w^G}} := R_w^G(1) = \sum_{i\in\mathbb{Z}} (-1)^i H_c^i(X(w), \overline{\mathbb{Q}}_\ell).$$

Its irr constituents are called <u>unipotent</u> rep, a very important family of reps.

relates to a deep thm of Lusztig

<u>Example</u> (Harish-Chandra induction)

Let $W_I \subseteq W$ be a parabolic subgp.
Let $P_I := B W_I B$ be the corresponding parabolic of G
Let L_I be the corresponding Levi of P_I.

Assume that W_I is F-stable and $w \in W_I$.

Then $$R_w^G(\theta) = \underbrace{\mathrm{Ind}_{P_I^F}^{G^F} \circ \mathrm{Inf}_{L_I^F}^{P_I^F}}_{\text{Harisch-Chandra induction}} \left(R_w^{L_I}(\theta)\right)$$

In particular, if $w=1$, then
$$R_1^G(\theta) = \mathrm{Ind}_{B^F}^{G^F} \mathrm{Inf}_{T^F}^{B^F} \theta .$$

Also note that $X(1) = G^F/B^F$, $\widetilde{X}(1) = G^F/U^F$

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(2): Uniform functions

<u>Definition</u>: A class function on G^F is uniform if it is
(Kilmoyer?) a $\overline{\mathbb{Q}}_\ell$-linear **combination** of DL-characters.

<u>Theorem</u>: The regular character of G^F is uniform:
$$\mathrm{reg}_{G^F} = \frac{1}{|W|} \sum_{w \in W} \dim R_w^G \cdot R_w^G(\mathrm{reg}_{T_w^F})$$

<u>Coro</u>: Every irr character of G^F occurs in some $R_w^G(\theta)$.

Proposition: The trivial rep is uniform:

$$1_{G^F} = \frac{1}{|W|} \sum_{w\in W} R_w^G$$

Proof: The Bruhat decomposition implies $G/B = \coprod_{w\in W} X(w)$.

$\Rightarrow \sum_i (-1)^i H_c^i(G/B, \bar{\mathbb{Q}}_\ell) = \sum_{w\in W} R_w^G$ is a trivial rep as $G \circlearrowright H_c^i(G/B, \bar{\mathbb{Q}}_\ell)$ trivially. It is of dim $= |W|$ (by Schubert cell decomposition). ▨

Proposition: The characteristic functions on s.s. classes are uniform.

Remark: Not all class functions are uniform e.g. some irr characters of $SL_2(\mathbb{F}_q)$ are not.

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(3) Orthogonality relations

Let: $w, w' \in W$, $\theta \in \mathrm{Irr}\, T^{wF}$, $\theta' \in \mathrm{Irr}\, T^{w'F}$

Theorem We have

$$\langle R_w^G(\theta), R_{w'}^G(\theta') \rangle_{G^F} = \#\left\{ v\in W \,\middle|\, \underbrace{w' = vwF(v)^{-1}}_{F\text{-conjugation}},\ \theta' = {}^v\theta \right\}$$

Definition: θ is said to be in general position if $C_W(wF, \theta) := \{v\in W \mid w = vwF(v)^{-1},\ \theta = {}^v\theta\} = 1$

Coro: If θ is in general position, then $\pm R_w^G(\theta)$ is irr.

<u>Example</u>: $G = GL_n$, $T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$

$\boxed{\text{Case-I}}$: $w = (1\ 2\ \cdots\ n)$ a Coxeter element

Then $T^{wF} = \left\{ \begin{pmatrix} \lambda & & & \\ & \lambda^q & & \\ & & \ddots & \\ & & & \lambda^{q^{n-1}} \end{pmatrix} \,\middle|\, \lambda \in \mathbb{F}_{q^n}^\times \right\} \cong \mathbb{F}_{q^n}^\times$

Fix a gp bijection $\operatorname{Irr} T^{wF} \xleftrightarrow{\sim} \mathbb{F}_{q^n}^\times$

$\rightsquigarrow$ θ is in general position iff

$\theta \mapsto$ an element in $\mathbb{F}_{q^n}^\times \setminus \bigcup_{\substack{d \mid n \\ d \neq n}} \mathbb{F}_{q^d}^\times$

In this case $(-1)^{n+1} \cdot R_w^G(\theta)$ is irreducible

$\boxed{\text{Case-II}}$: $w = 1$ the identity.

Then $T^{wF} = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \,\middle|\, \lambda_i \in \mathbb{F}_q^\times \right\} \cong (\mathbb{F}_q^\times)^n$

$\Rightarrow \theta = \theta_1 \times \cdots \times \theta_n$, $\theta_i \in \operatorname{Irr} \mathbb{F}_q^\times$

$\rightsquigarrow$ θ is in general position iff

$\theta_i \neq \theta_j \quad \forall\ i \neq j$

If $n \geq q$, this is never going to happen.

(4) Lusztig series

w & w' not F-conjugate

$\xRightarrow[\text{orthogonality}]{} \langle R_w^G(\theta), R_{w'}^G(\theta') \rangle_{G^F} = 0.$

However, as $R_w^G(\theta)$ & $R_{w'}^G(\theta)$ are virtual, they may still have common irr constituents.
($\langle a+b, a-b \rangle_{G^F} = 0$)

⇝ **Lusztig series**

Fact:

There is a "natural" map (depending on $\mathbb{F}_q^\times \cong (\mathbb{Q}/\mathbb{Z})_{p'} \ldots$)

$$\{\text{pairs } (w, \theta)\} \longrightarrow \{\text{s.s. elements in } G^*\}$$

G^* is the Langlands dual of G over $\mathbb{F}_q$.
e.g. $GL_n^* = GL_n$, $PGL_n^* = SL_n$ ($Sp_{2n}^* = SO_{2n+1}$)

Definition Let (s) be a s.s. class in G^{*F^*}.

$$\mathcal{E}(G^F, (s)) := \{\text{irr constituents of } R_w^G(\theta) \mid (w, \theta) \mapsto (s)\}$$

is called the Lusztig series associated to (s).

<u>Example</u>: Take $G^F = PGL_2(\mathbb{F}_5)$ $(\cong S_5)$

uniprep. true in general — $\{1, St\} \longleftrightarrow 1 \in SL_2(\mathbb{F}_5)$

$\{sgn, sgn \otimes St\} \longleftrightarrow -1 \in SL_2(\mathbb{F}_5)$

generic cusp — $\{\rho_4\} \longleftrightarrow$ a regular s.s. of order 6

$\{sgn \otimes \rho_4\} \longleftrightarrow$ another regular s.s. of order 6.

generic principal series — $\{\rho_6\} \longleftrightarrow$ a reg. s.s. of order 4.

(If $sgn(x) = -1$, then $\rho_6(x) = 0$, so $\rho_6 = sgn \otimes \rho_6$.)

<u>Theorem</u>. (For simplicity, assume $Z(G) = Z(G^\circ)$)

(i): $\mathrm{Irr}\, G^F = \coprod_{(s)} \mathcal{E}(G^F, (s))$

(ii): $\mathcal{E}(G^F, (s)) \xleftrightarrow{1:1} \mathcal{E}(C_{G^{*F^*}}(s)^F, 1)$

a rather deep theorem of Lusztig, main result in a 300 pages monograph.

<u>Example</u>: $G = GL_n \Rightarrow G^* = GL_n$

$\Rightarrow C_{G^{*F^*}}(s)^F$ is a direct prod of GL_i's,

$(= \prod GL_{n_i}(\mathbb{F}_{q^{d_i}}),\ \Sigma n_i d_i = n)$

$\rightsquigarrow$ a large part of the study of $\mathrm{Rep}(GL_n(\mathbb{F}_q))$ is reduced to unipotent reps.