3D plabic graphs & propagation rules

Last lecture: Deodhar torus $T_\beta = \{ F^0 \xrightarrow{\delta(\beta_j)} F^j \ \forall j \} \subseteq X(\beta)$

$$\parallel$$

$$\{ \Delta_j := \Delta_{\delta(\beta_j)[i_j],[i_j]}(B_{\beta_j}(z_1,\dots,z_j)) \neq 0 \quad \text{where } j \in J_\beta \}$$

chamber minors

j s.t. $\delta(\beta_{j-1}) = \delta(\beta_j)$ "solid" crossings

- Expectation: this is a cluster torus. What is our guess for the cluster variables?
- We have the right # of chamber minors, since $|J_\beta| = r - \binom{n}{2} = \dim X(\beta)$.
  But they are not irred, so can't be cluster var.

  Our guess for cluster var is (essentially) the irred. factors of chamber minors.
- Understand factors of Δ_j by analyzing behavior on Deodhar hypersurfaces $\{V_j'\}_{j\in J_\beta}$.
  Add: why this helps, intuition for order of vanishing

Lemma: $d \in J_\beta$, $j \in \{1,\dots,r\}$. Order of vanishing of Δ_j along V_d' is

$$\operatorname{ord}_{V_d'} \Delta_j = \begin{cases} 1 & \text{if } \delta(\beta_j)\cdot\{1,\dots,i_j\} \neq \omega_j^{\langle d\rangle}\cdot\{1,\dots,i_j\} \\ 0 & \text{else} \end{cases}$$

seq. of perm. equal to $\delta(\beta_i)$ for $i<d$. Then make "mistake" at $i=d$, then do Dem. prod. after.

e.g. $\beta = (\overset{1}{1},\overset{2}{1},\overset{3}{2},\overset{4}{2},\overset{5}{1},\overset{6}{1})$

$$\begin{array}{lcccccc}
i_j & & 1 & & 2 & & 1 \\
\delta(\beta_j) = (& s_1, & \overset{213}{s_1}, & s_1s_2, & \overset{231}{s_1s_2}, & s_1s_2s_1, & \overset{321}{s_1s_2s_1}) \\
\omega_j^{\langle 2\rangle} = (& s_1, & \overset{123}{e}, & s_2, & \overset{132}{s_2}, & s_2s_1, & \overset{312}{s_2s_1}) \\
\omega_j^{\langle 4\rangle} = (& , & , & , & \overset{213}{s_1}, & s_1, & \overset{213}{s_1}) \\
\omega_j^{\langle 6\rangle} = (& , & , & , & , & , & \overset{231}{s_1s_2})
\end{array}$$

$J_\beta = \{2,4,6\}$

$\Delta_2 = z_2$

$\Delta_4 = z_2 z_4$

$\Delta_6 = z_4 z_6$

Defn of (potential) cluster variables

Consider mtx

$$J_\beta \left\{ d \left[\overbrace{\quad \square \quad}^{J_\beta} \right] =: M \right. \qquad \square = \operatorname{ord}_{V'_d} \Delta_j \text{ (column } j\text{)}$$

e.g.

$$V_{d'} \quad \begin{array}{c|ccc} d \backslash j & 2 & 4 & 6 \\ \hline 2 & 1 & 1 & 0 \\ 4 & 0 & 1 & 1 \\ 6 & 0 & 0 & 1 \end{array} = M \qquad (\text{columns } \Delta_j)$$

Cor: M is a uni-upper triangular integer mtx & so $M^{-1} = (a_{dj})$ uni-upper triangular integer matrix

Pf: If $j<d$, $\partial(\beta_j) = w_j^{\langle d \rangle}$, so Lem. $\Rightarrow$ $\operatorname{ord}_{V_d'} \Delta_j = 0$

$j=d$, $\partial(\beta_j) = w_j^{\langle d \rangle} s_{i_j}$, so Lem $\Rightarrow$ $= 1$

} M is upper-triangular w/ 1's on diag.

Defn: $d \in J_\beta$. $\quad x_d := \prod_{j \in J_\beta} \Delta_j^{a_{jd}}$

↑ use col d of M^{-1} to give exponents on Δ_j.

e.g.

$$M^{-1} = \begin{array}{c|ccc} j \backslash d & 2 & 4 & 6 \\ \hline 2 & 1 & -1 & 1 \\ 4 & & 1 & -1 \\ 6 & & & 1 \end{array}$$

$x_2 = \Delta_2 = z_2 \qquad x_4 = \Delta_2^{-1}\Delta_4 = \frac{z_2 z_4}{z_2} = z_4 \qquad x_6 = \Delta_2 \Delta_4^{-1} \Delta_6 = z_2 \cdot \frac{1}{z_2 z_4} \cdot z_4 z_6 = z_6.$

Note: $\Delta_j = \prod_{d \in J_\beta} x_d^{m_{dj}}$ by construction

↑ use col j of M to give exponents on x_d

Can use 3D plabic graphs to get M.

Prop: For $d \in J_\beta$, x_d is an irreducible regular fcn on $X(\beta)$ & vanishes to order

$$\begin{cases} 1 & \text{on } V_d' \\ 0 & \text{on } V_i' \quad i \neq d \end{cases}.$$

Also, $T_\beta = \{ x_d \neq 0 \ \ \forall d \in J_\beta \}$.

Pf: Order of vanishing is by construction. $\operatorname{ord}_{V_i'} x_d = \sum_j a_{jd} \underbrace{\operatorname{ord}_{V_i'} \Delta_j}_{m_{ij}} = (i,d)$ entry of $M \cdot M^{-1}$

A priori, x_d is rat'l. But x_d non-vanishing on $T_\beta \Rightarrow$ regular on T_β. Ord of vanishing formula $\Rightarrow$ x_d regular on $\bigcup_{\substack{i \in J_\beta \\ \text{mutable}}} V_i$ $\Rightarrow$ x_d regular on

$$X(\beta) = T_\beta \cup \bigcup_{\substack{i \in J_\beta \\ \text{mutable}}} V_i .$$

Irreducibility: the locus $\{x_d = 0\} \subseteq X(\beta)$ is exactly V_d, which is irreducible.
(need modification of this argument for froz. d)

$T_\beta = \{x_d \neq 0 \ \ \forall d \in J_\beta\}$ follows from formula for $\{\Delta_j\}_{j \in J_\beta}$ in terms of $\{x_d\}$. ∎

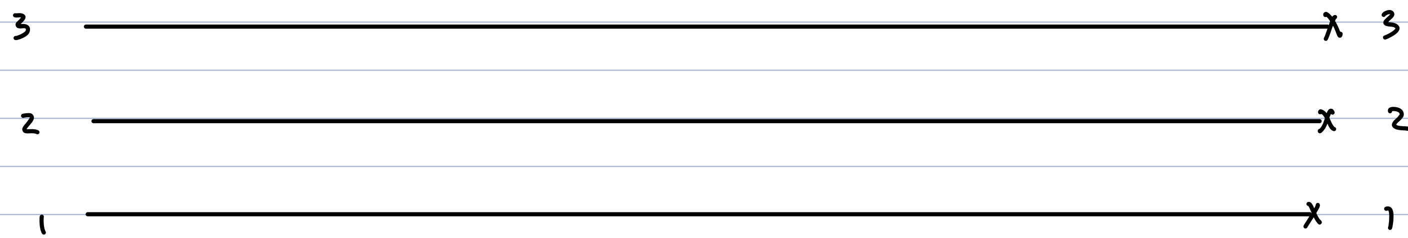
$$\boxed{\text{Our guess for a cluster is } \{x_j\}_{j \in J_\beta}}$$

Now the question is: what is the quiver? We'll spend the rest of the lecture on 3D plabic graphs, from which we'll extract quiver on Thurs.

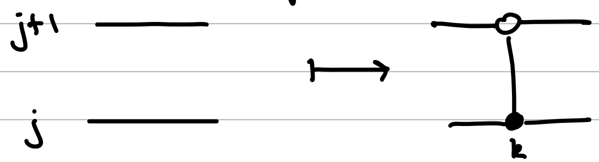
3D plabic graphs : combin. way to encode grid minors & their factorization.

To construct G_β for $\beta \in Br_n^+$ (e.g. $\beta = (1,1,2,2,1,1,2,1) \in Br_3^+$)

① Start w/ n horiz wires

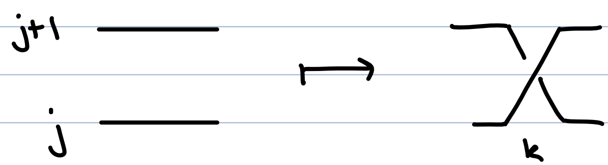


② Reading β left-to-right, for letter $i_k = j$ in β, do local replacement

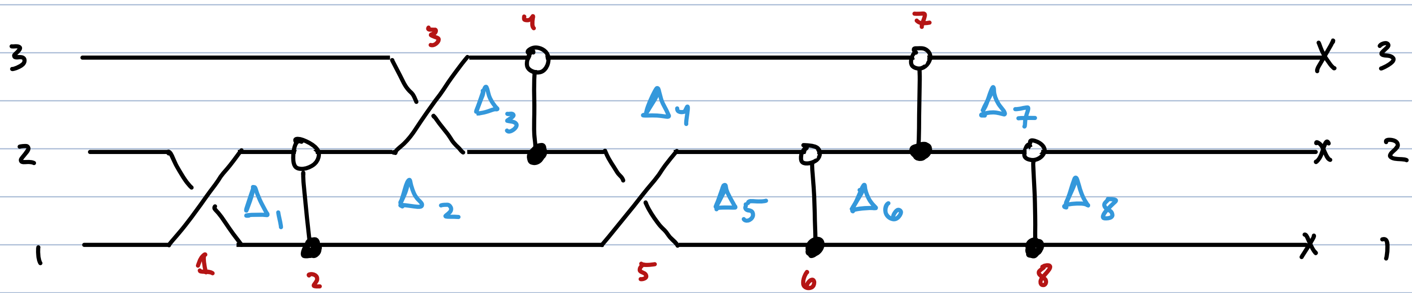


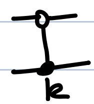
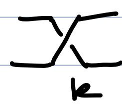
if k solid

&



if k hollow

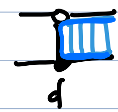


• We often label region right of  or  w/ Δ_k.

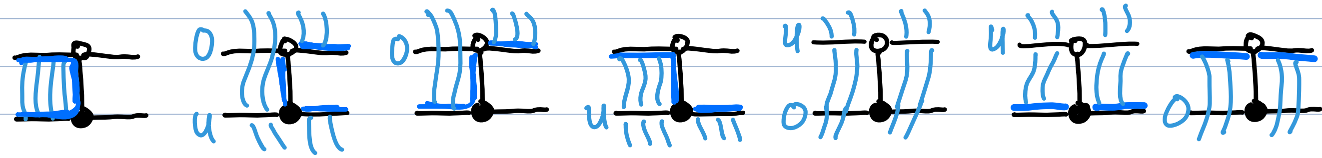
mention here - certain config corresp. to 3-term reln among minors?

Now, we want to understand factorization $\Delta_j = \prod_{d \in J_\beta} x_d^{m_{dj}}$ in terms of G_β.

Defn: For $d \in J_\beta$, define a "soap film" S_d (= union of regions of G_β) as follows:

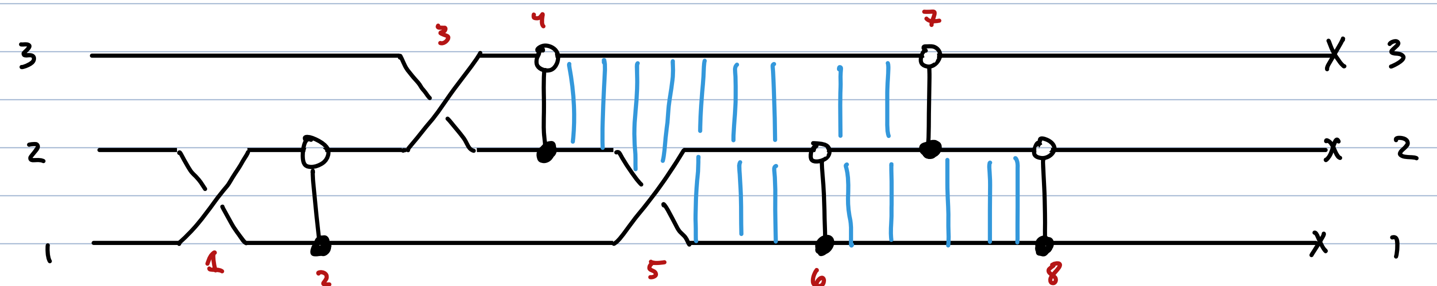
S_d starts like  (clinging to horiz wires R of d^{th} bridge)

& propagates to right according to



S_d over top strand OR under bottom strand $\Rightarrow$ bridge part of ∂S_d.

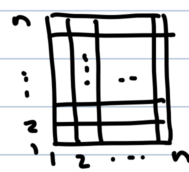
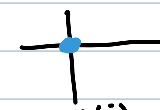
e.g.



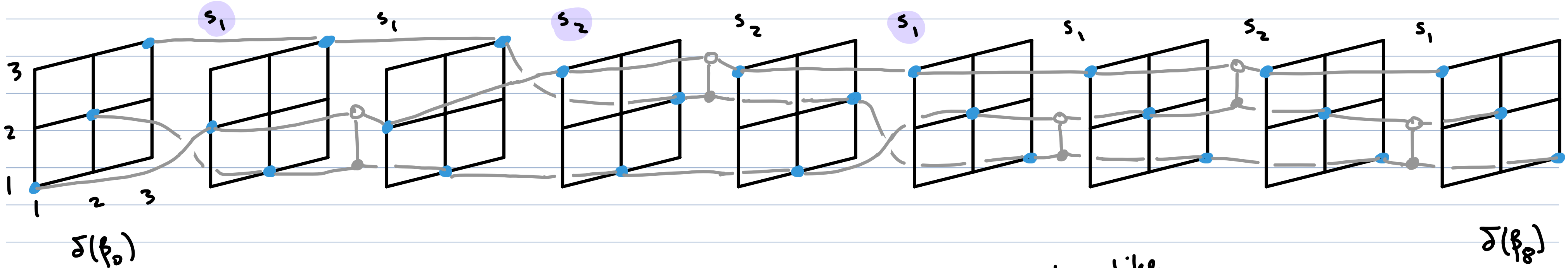
Prop: Let $d \in J_\beta$, $j \in [r]$. Then x_d divides Δ_j $\Leftrightarrow$ Δ_j vanishes on V_d' $\Leftrightarrow$ S_d covers region labeled Δ_j in G_β.

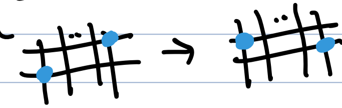
So in ex above, x_4 divides $\Delta_4, \Delta_5, \Delta_6$ & no others chamber minors.

Pf sketch:

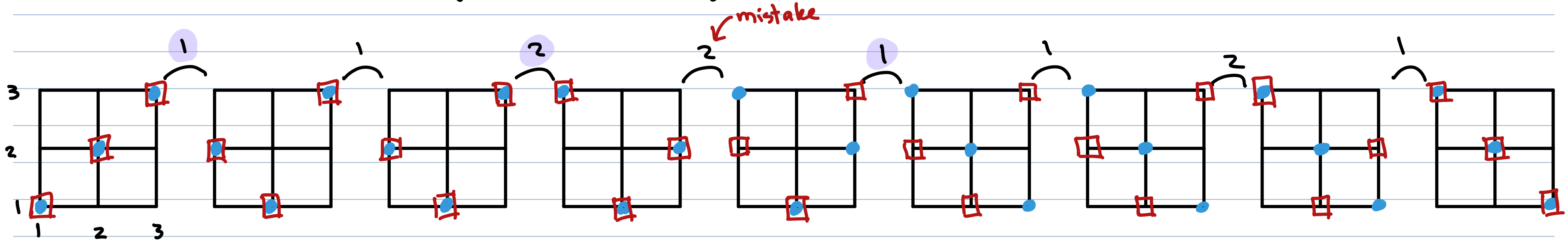
$v \in S_n$. Diagram for v: In grid , draw • on i 

We view G_β as being drawn between diag. for $\delta(\beta_0), \cdots, \delta(\beta_r)$

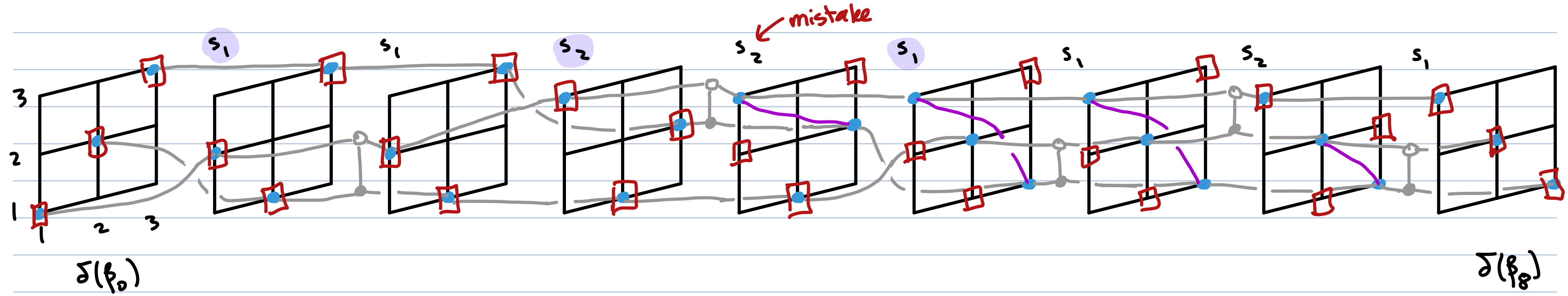


At hollow crossing i_k, swap rows i_k, i_k+1 of diagram. [Looks like  O/w, do nothing.

We can also draw diagrams for $\omega_j^{\langle d\rangle}$ on top.

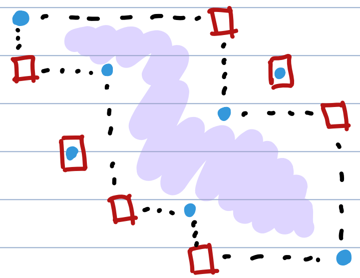


If we know $\delta(\beta_j)$ & $\omega_j^{\langle d\rangle}$, can determine if $x_d \mid \Delta_j$. But we can actually use less info.



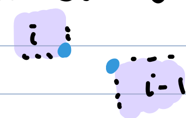
Summary: Just need to track rows where col pos of •, □ don't match. Can do this w/ curves. Evolution of curves is constrained bc. •, □ evolve according to Dem. prod.

Lemma: Let $\delta_j := \delta(\beta_j)$ & $\omega_j := \omega_j^{(d)}$. Draw a line from ea. ● to the □ in same row & col, and a line from ea. □ to the ● in same row & col.

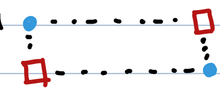


Then we obtain a collection of disjoint skew-shapes (in French not.) where "outer corners" are □, "inner corners" are ● & NW & SE corners are ●. (+ some ⊡ outside of the shape)

Ordering skew shapes by row of SE corner, shape i is strictly NW of shape $i-1$.

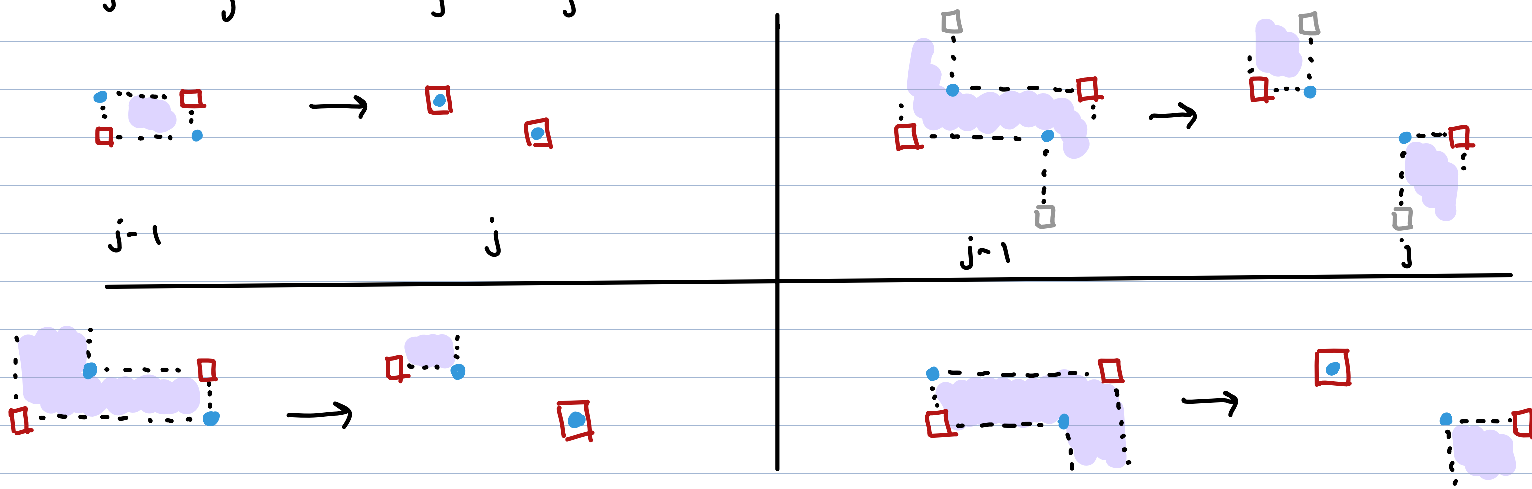


Pf: True for $j=1,\dots,d-1$. For $j=d$, get

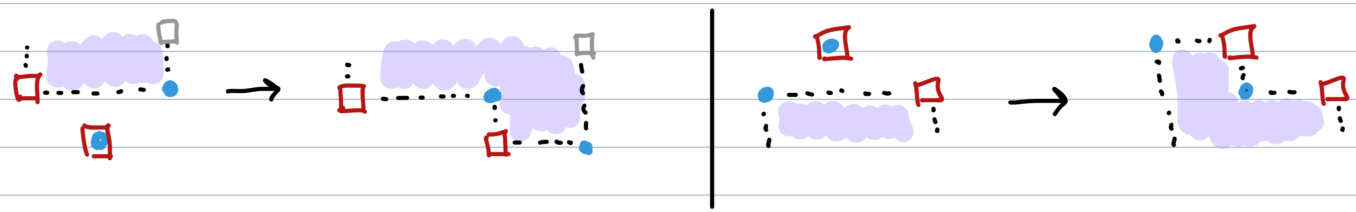


Then proceed by induction, checking cases (won't do all): 1) $\delta_{j-1} = \delta_j$ & $\omega_{j-1} = \omega_j$

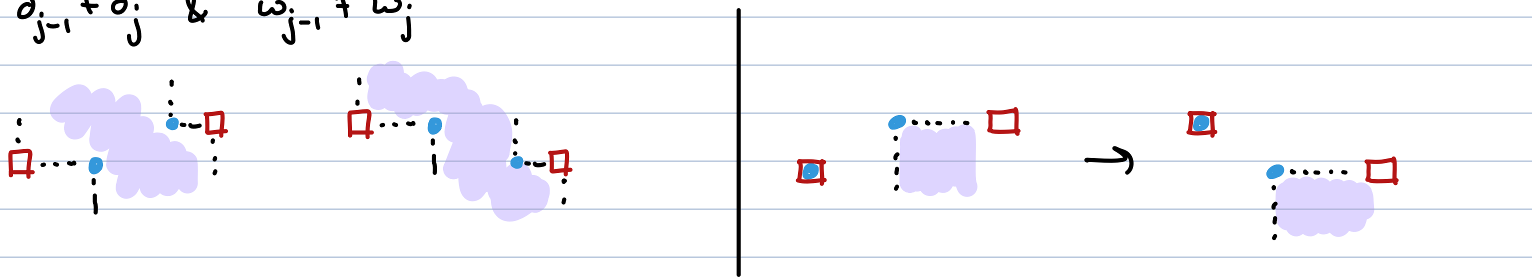
2) $\delta_{j-1} = \delta_j$ & $\omega_{j-1} \neq \omega_j$



3) $\delta_{j-1} \neq \delta_j$ & $\omega_{j-1} = \omega_j$

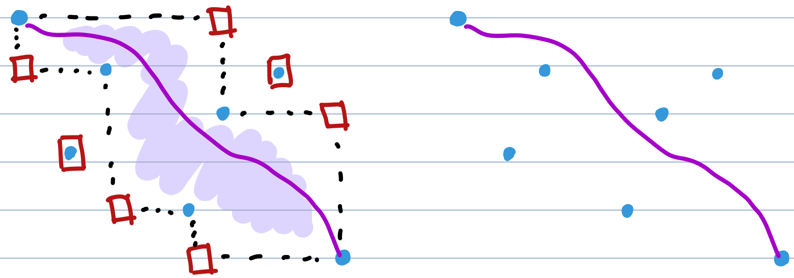


4) $\delta_{j-1} \neq \delta_j$ & $\omega_{j-1} \neq \omega_j$



& similar

<u>Note</u>: Skew shapes from lemma are determined by their NW, SE & inner corners.



We record this by drawing a curve from NW to SE corner. Inner corners = closest • to curve.

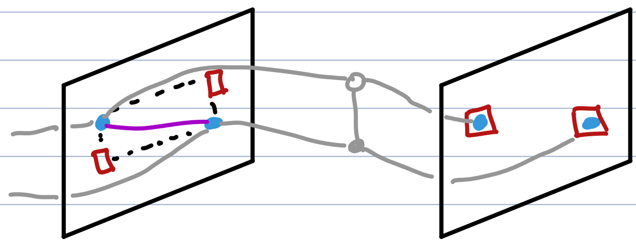
<u>Note</u>: $\delta_j[a] = \omega_j[a]$ $\Longleftrightarrow$ the grid line $y=a$ does not intersect the interior of any skew-shape curve

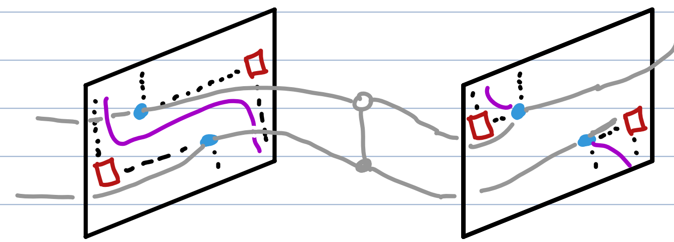
$\Longleftrightarrow$ ______ " ________________

$\delta_j[a] \neq \omega_j[a]$ $\Longleftrightarrow$ the grid line $y=a$ intersects the interior of some curve.

Now, we draw the curves on the diagrams & observe how they change in different scenarios. Curve going under/over a strand in the projection just records which side of it's on in the diagram.

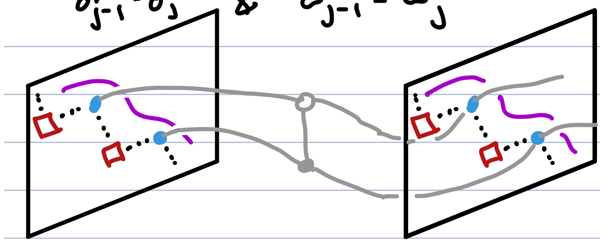
$\delta_{j-1} = \delta_j$ & $\omega_{j-1} \neq \omega_j$





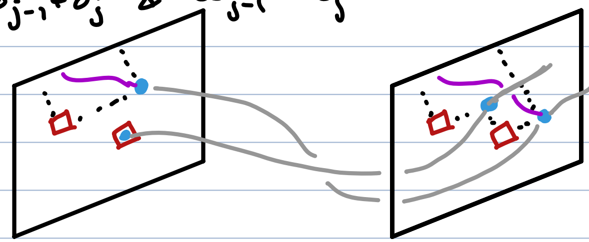
curve over top or under bottom gets cut

$\delta_{j-1} = \delta_j$ & $\omega_{j-1} = \omega_j$



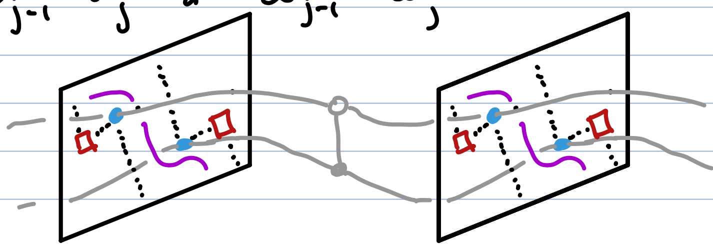
curve goes under undisturbed

$\delta_{j-1} \neq \delta_j$ & $\omega_{j-1} = \omega_j$



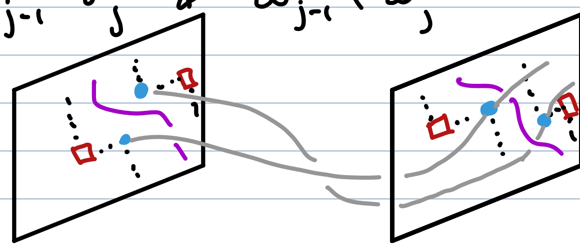
curve follows strand it ended on

$\delta_{j-1} = \delta_j$ & $\omega_{j-1} = \omega_j$



curve under top or over bottom undisturbed

$\delta_{j-1} \neq \delta_j$ & $\omega_{j-1} \neq \omega_j$



These are the propagation rules!