Lecture 3.- Deodhar torus

Let $\beta = \sigma_{i_1} \cdots \sigma_{i_r}$ be a braid word. Recall that the braid variety is

$$X(\beta) := \{ (\mathcal{F}^0, \ldots, \mathcal{F}^r) \mid \mathcal{F}^{std} = \mathcal{F}^0 \xrightarrow{s_{i_1}} \mathcal{F}^1 \xrightarrow{s_{i_2}} \cdots \xrightarrow{s_{i_r}} \mathcal{F}^r = \mathcal{F}^{ant} \}$$

For $j = 1, \ldots, r$, we'll consider the partial braid words

$$\beta_j = \sigma_{i_1} \cdots \sigma_{i_j}.$$

Note that, by the definition of the Demazure product:

$$\mathcal{F}^0 \xrightarrow{w_j} \mathcal{F}^j \quad \text{with} \quad w_j \leq \delta(\beta_j).$$

Def The Deodhar torus $T_\beta \subseteq X(\beta)$ is

$$T_\beta := \{ (\mathcal{F}^0, \ldots, \mathcal{F}^r) \in X(\beta) \mid \mathcal{F}^0 \xrightarrow{\delta(\beta_j)} \mathcal{F}^j \ \forall j = 1, \ldots, r \}$$

Note T_β depends on the chosen braid word β, and not just on the braid β.

Lemma The Deodhar torus is indeed an open torus in $X(\beta)$.

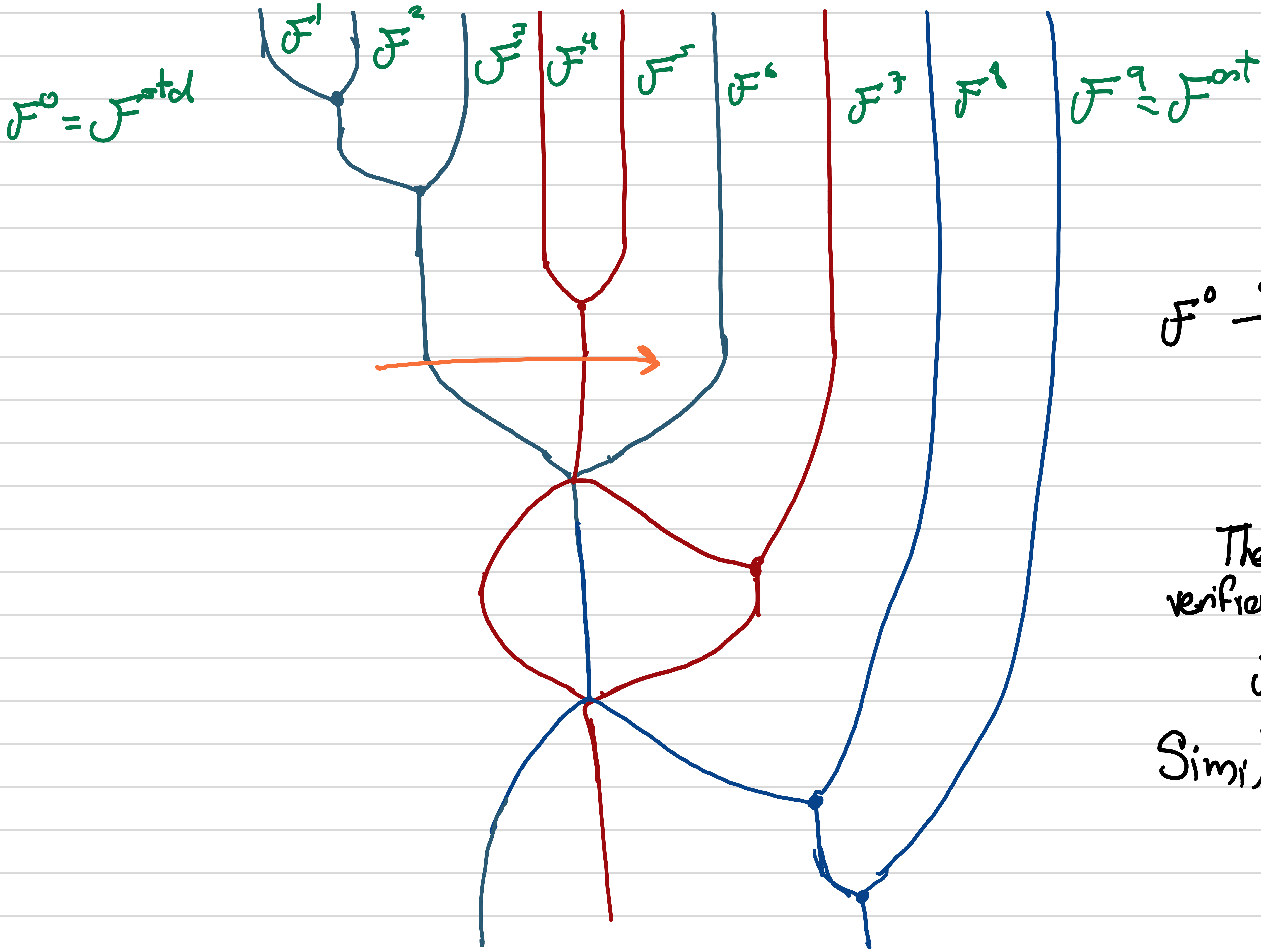
To prove this lemma, we construct a Demazure weave $\mathfrak{w}$ on β whose associated torus is T_β.

This is the left-to-right inductive weave $\vec{\mathfrak{w}}_\beta$.

Def We define the weave $\vec{\mathfrak{w}}_\beta$ inductively on the length of β.

- If $\beta = \sigma_i$, then $\vec{\mathfrak{w}}_\beta = \;|$
- If $\beta = \beta' \sigma_i$ we have two options.
  - If $\delta(\beta) = \delta(\beta') s_i$, then $\vec{\mathfrak{w}}_\beta = \boxed{\vec{\mathfrak{w}}_{\beta'}}\;|$
  - If $\delta(\beta) = \delta(\beta')$, then $\vec{\mathfrak{w}}_\beta = \boxed{\vec{\mathfrak{w}}_{\beta'}}$

Example: $\beta = 111221211$



Let us verify that

$$\mathcal{F}^0 \xrightarrow{\delta(\beta_5)} \mathcal{F}^5 \quad \forall\, (\mathcal{F}^0, \dots, \mathcal{F}^9) \in T_{\vec{\beta}}$$

Note that $\delta(\beta_5) = \delta(11122)$
$= 12$

The orange arrow $\rightarrow$ tha verifies that

$$\mathcal{F}^0 \xrightarrow{s_1 s_2} \mathcal{F}^5.$$

Similarly, $\mathcal{F}^0 \xrightarrow{\delta(\beta_i)} \mathcal{F}^i \;\; \forall\, i.$

So the Deodhar torus $T_{\underline{\beta}}$ coincides with $T_{\vec{w}_{\underline{\beta}}}$.

Let us obtain candidates for the cluster var-s. associated to this (tentative) cluster torus.

Assume $(\mathcal{F}^0, \dots, \mathcal{F}^r) \in T_{\underline{\beta}}$, then

$$\mathcal{F}^{std} = \mathcal{F}^0 \xrightarrow{\delta(\underline{\beta}_{j})} \mathcal{F}^j \quad \forall j = 1, \dots, r.$$

Now, we have $\mathcal{F}(\delta(\underline{\beta}_j) w_0) \xrightarrow{w_0 \delta(\underline{\beta}_j)^{-1}} \mathcal{F}^{std} \xrightarrow{\delta(\underline{\beta}_j)} \mathcal{F}^j \Rightarrow \mathcal{F}(\delta(\underline{\beta}_j) w_0) \xrightarrow{w_0} \mathcal{F}^j$

i.e, $\mathcal{F}(\delta(\underline{\beta}_j) w_0)$ and $\mathcal{F}^j$ are transverse $\forall j$.

Transversality can be written as a non-vanishing condition!

<u>Lemma</u> $\mathcal{F}(w w_0) \pitchfork \mathcal{F}(M)$ iff $\forall i = 1, \dots, n$, $\Delta_{w[i],[i]}(M) \neq 0$, where $[i] = \{1, \dots, i\}$

$w[i] = \{w(1), \dots, w(i)\}$

<u>Pf Sketch</u> Let $M = (m_1 | m_2 | \cdots | m_n)$

$$\mathrm{span}\{m_1\} \cap \mathrm{span}\{e_{ww_0(1)}, \dots, e_{ww_0(n-1)}\} = \mathrm{span}\{m_1\} \cap \{e_{w(n)}, \dots, e_{w(2)}\} = \{0\}$$

$\Leftrightarrow \Delta_{w(1),1}(M) \neq 0.$

$\mathrm{span}\{m_1, m_2\} \cap \mathrm{span}\{e_{w(n)}, \dots, e_{w(3)}\} = \{0\} \Leftrightarrow \Delta_{\{w(1),w(2)\},\{1,2\}}(M) \neq 0$, and so on $\square$

Now, recalling $F^j = F(B_{\beta_j}(z_1, \dots, z_j))$ we obtain that, if $(F^0, \dots, F^r) \in T_\beta$ then

$$(\star) \qquad \Delta_{\delta(\beta_j)[i],[i]}(B_{\beta_j}(z_1, \dots, z_j)) \neq 0 \quad \forall j = 1, \dots, r, \ \forall i = 1, \dots, n.$$

Great, we have obtained regular functions on $X(\beta)$ nowhere vanishing on T_β.

<u>Problem</u> These are too many!!! rn of them. Ideally, we want $\dim(X(\beta)) = r - \binom{n}{2}$ of them

Luckily, the list $(\star)$ is <u>very</u> redundant. As you'll see in the exercises, the minors of M and of $MB_j(z)$ are very similar. Think of $M = I$: $\Delta_{[i],[i]} B_j(z) = 1 = \Delta_{[i],[i]} I$ <u>unless</u> $i = j$.

So, at each step j we don't need to look at <u>all</u> the minors $\Delta_{\delta(\beta_j)[i],[i]} B_{\beta_j}(z_1, \dots, z_j)$.
There is only one new one among them, obtained precisely when $i = i_j$.

<u>Def</u> The <u>grid minors</u> of β are $\Delta_{\delta(\beta_j)[i_j],[i_j]} B_{\beta_j}(z_1, \dots, z_j)$, $j = 1, \dots, r$.

There is still some redundancy. If $\delta(\beta_j \sigma_{i_{j+1}}) = \delta(\beta_j) s_{i_{j+1}}$, then

$$\mathcal{F}^0 \xrightarrow{\delta(\beta_j)} \mathcal{F}^j \text{ automatically implies that } \mathcal{F}^0 \xrightarrow{\delta(\beta_{j+1})} \mathcal{F}^{j+1}.$$

So we don't need this latter condition.

<u>Def</u> We say that j is a solid crossing of β if $\delta(\beta_{j-1}) = \delta(\beta_j)$
hollow crossing if $\delta(\beta_{j-1}) \neq \delta(\beta_j)$

$J_\beta :=$ set of solid crossings

<u>Def</u> The <u>chamber minors</u> of β are $\Delta_{\delta(\beta_j)[i_j],[i_j]} B_{\beta_j}(z_1, \dots, z_j)$, j is solid

So, chamber minors $\subseteq$ grid minors.

The chamber minors are our candidates for cluster variables. But they don't quite work, because they don't need to be irreducible.

Example. Let $\beta = \underset{\text{Hollow}}{1}\ \underset{\text{Solid}}{1}\ \underset{H}{2}\ \underset{S}{2}\ \underset{H}{1}\ \underset{S}{1}$. The chamber minors are:

$$\Delta_{\{2,3\},\{1\}} B_{\sigma_1\sigma_1}(z_1,z_2) = z_2$$

$$\Delta_{s_1 s_2[2],[2]} B_{\sigma_1^2\sigma_2^2}(z_1,z_2,z_3,z_4) = \Delta_{\{2,3\},\{1,2\}} B_{\sigma_1^2\sigma_2^2}(z_1,\dots,z_4) = z_2 z_4$$

$$\Delta_{\{3\},\{1\}} B_\beta(z_1,\dots,z_6) = z_4 z_6$$

These are not irreducible. How do we fix this?

Assume that there is indeed a cluster structure on $X(\beta)$ s.t. T_β is a cluster torus. Then, $X(\beta)\setminus T_\beta$ should be a union of irred. hypersurfaces, one per mutable variable (we'll see how to deal with frozens in a bit)

Since Δ_j is nonvanishing on T_β, Δ_j should be expressible as a monomial in the (hopefully existent) cluster variables

$$\Delta_j = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$$

For each hypersurface $V_k = \{x_k = 0\}$, we want to determine α_k (note that $\alpha_k \geq 0$ if x_k is mutable b/c Δ_j is a regular function). Formally speaking, α_k is the order of vanishing of Δ_j along V_k.

So let's define our candidate hypersurfaces.

Def Let $j \in J_\beta$. We define a sequence of elements $w_1^{\langle j \rangle}, \dots, w_r^{\langle j \rangle} \in S_n$ as follows.

$$w_1^{\langle j \rangle} = \delta(\beta_1), \dots, \quad w_{j-1}^{\langle j \rangle} = \delta(\beta_{j-1}), \quad w_j^{\langle j \rangle} = \delta(\beta_{j-1}) s_{i_j}, \quad w_{j+1}^{\langle j \rangle} = \begin{cases} w_j^{\langle j \rangle} s_{i_{j+1}}, & \text{if } w_j^{\langle j \rangle} s_{i_{j+1}} > w_j^{\langle j \rangle} \\ w_j^{\langle j \rangle}, & \text{else} \end{cases}$$

$$W^{\langle j\rangle}_{j+t} = \begin{cases} W^{\langle j\rangle}_{j+t-1} s_{i_{j+t}}, & \text{if } W^{\langle j\rangle}_{j+t-1} s_{i_{j+t}} > W^{\langle j\rangle}_{j+t-1} \\ W^{\langle j\rangle}_{j+t-1}, & \text{else} \end{cases}$$

In other words, to obtain $W^{\langle j\rangle}_t$, compute the Demazure product inductively but make a mistake at the j-th step.

Def Let $j \in J_\beta$. The <u>Deodhar hypersurface $V_j \subseteq X(\beta)$ is</u> closure

$$V_j := \{(\mathcal{F}^{std}, \dots, \mathcal{F}^r = \mathcal{F}^{ant}) \in X(\beta) \mid \mathcal{F}^{std} \xrightarrow{W^{\langle j\rangle}_t} \mathcal{F}^t \ \forall t\}$$

The Deodhar hypersurface V_j, when nonempty, is irreducible and has codim 1 in $X(\beta)$

Example Let $\beta = 11\beta'$, $j=2$. Note that $V_2 \neq \emptyset$ iff $\delta(\beta') = w_0$, and in this case $V_2 = \mathbb{C} \times X(\beta')$. More generally, if $\beta = \beta_1 1 \underset{j}{1} \beta_2$ then $V_j = \mathbb{C} \times X(\beta_1 \beta_2)$.

WARNING The way it's defined, V_j may be $\emptyset$ if $w_r^{\langle j \rangle} \neq w_0$ (this happens, for example, if $r \in J_\beta : V_r = \emptyset$).

We can define the Deodhar hypersurface to be $\neq \emptyset$ if we forget about the condition $\mathcal{F}^r = \mathcal{F}^{ant}$. Note that the variety of chains of flags without this condition is simply $\mathbb{G}^r$. Let us denote by $V_j' \subseteq \mathbb{G}^r$ the Deodhar hypersurface defined this way. Note that

$$V_j' \cap X(\beta) = V_j \quad \text{if} \quad w_r^{\langle j \rangle} = w_0.$$ We'll call this a mutable Deodhar hypersurface

$$V_j' \cap X(\beta) = \emptyset \quad \text{if} \quad w_r^{\langle j \rangle} < w_0.$$ We'll call this a frozen Deodhar hypersurface

The way we have defined them, the chamber minors are polynomials in $z_1, \dots, z_r$. So it still makes sense to talk about the order of vanishing of Δ_j along V_k'.

Lemma Let $k \in J_\beta$ be a solid crossing, and $j = 1, \dots, r$ any crossing. The order of vanishing of Δ_j along V'_k is 0 or 1.

Pf The Deodhar hypersurface V'_d is cut out by the vanishing of a minor, that has degree 1 on some parameter $z_{i(k)}$. But by Cauchy-Binet Δ_j has degree at most 1 in each of the parameters $z_1, \dots, z_j$.

More precisely:

Lemma The order of vanishing of Δ_j along V'_k is

$$\operatorname{ord}_{V'_k} \Delta_j = \begin{cases} 1, & \text{if } \delta(\beta_j)[i_j] \neq W_j^{\langle k \rangle}[i_j] \\ 0, & \text{else} \end{cases}$$

Pf Note that $\Delta_j(\mathcal{F}^0,\dots,\mathcal{F}^j) = \Delta_{\delta(\beta_j)[i_j],[i_j]} B_{\beta_j}(z_1,\dots,z_j) \neq 0$ if and only if

$\mathcal{F}^j_{i_j} \cap \mathrm{span}\{e_p \mid p \notin \delta(\beta_j)[i_j]\} = \{0\}$. ($\heartsuit$)

Now, for a generic point in V'_k we have

$$\mathcal{F}^{std} \xrightarrow{w_j^{\langle k \rangle}} \mathcal{F}^j.$$

In particular, by an argument similar to the lemma on page 4 of this lecture, $\Delta_{w_j^{\langle k\rangle}[i],[i]}(B_{\beta}(z_1,\dots,z_j)) \neq 0$. Taking $i = i_j$, we see that

$\mathcal{F}^j_{i_j} \cap \mathrm{span}\{e_p \mid p \notin w_j^{\langle k\rangle}[i_j]\} = \{0\}$. Thus, we see that if $w_j^{\langle k\rangle}[i_j] = \delta(\beta_j)[i_j]$,

then $\mathrm{ord}_{V'_k} \Delta_j = 0$.

On the other hand, assume $w_j^{\langle k\rangle}[i_j] \neq \delta(\beta_j)[i_j]$. By definition, $\delta(\beta_j)$ is the largest (in Bruhat order) reduced subword of β_j, and $w_j^{\langle k\rangle}$ is a reduced subword of β_j. So $w_j^{\langle k \rangle} < \delta(\beta_j)$ in Bruhat order. This implies that $\forall i$, either $w_j^{\langle k \rangle}[i] = \delta(\beta_j)[i]$, or $\Delta_{\delta(\beta_j)[i],[i]}$ vanishes on $Bw_j^{\langle k \rangle}B$. But, by definition, if $(F^0, \dots, F^j)$ is a generic point then $F^0 \xrightarrow{w_j^{\langle k \rangle}} F^j$, ie, $B_{\beta_j}(z_1, \dots, z_j) \in Bw_j^{\langle k \rangle}B$. So, if $w_j^{\langle k \rangle}[i_j] \neq \delta(\beta_j)[i_j]$, Δ_j vanishes on a generic pt. of V_k'.