Tuesday, Lecture 1 || Locally acyclic cluster algebras

① Freezing and cluster localization

Recap: $\mathcal{A}$ = cluster algebra   Laurent phenomenon

$$\mathcal{U} = \bigcap_{\text{all seeds } s} \mathbb{C}[x_1^{(s)\pm}, \dots x_n^{(s)\pm}] \qquad \qquad \mathcal{A} \overset{\Downarrow}{\subset} \mathcal{U}.$$

Suppose $x_1^{(s)}, \dots x_j^{(s)}$ = mutable cluster variables in some cluster s.

~~We can~~ Let $\mathcal{A}'$ = cluster algebra with quiver Q' where $x_1^{(s)}, \dots x_j^{(s)}$ are frozen.

$\mathcal{U}'$ = resp. upper cluster algebra.

Lemma $\mathcal{A}' \underset{(1)}{\subset} \mathcal{A}[x_1^{(s)\pm}, \dots x_j^{(s)\pm}] \underset{(2)}{\subseteq} \mathcal{U}[x_1^{(s)\pm}, \dots x_j^{(s)\pm}] \underset{(3)}{\subseteq} \mathcal{U}'$

Pf (1) Mutation rules in $\mathcal{A}'$ are same as for $\mathcal{A}$, if vertex is not $x_1^{(s)} \dots x_j^{(s)}$. So all cluster variables in $\mathcal{A}'$ are a subset of cluster vars in $\mathcal{A}$, once we invert $x_1^{(s)}, \dots x_j^{(s)}$.

(2) is $\mathcal{A} \subset \mathcal{U}$; (3) follows from the fact that in $\mathcal{U}'$ we intersect less Laurent polynomial rings in same variables.

Cor: Assume $\mathcal{A}' = \mathcal{U}'$. Then $\mathcal{A}' = \mathcal{A}[x_1^{(s)\pm}, \dots x_j^{(s)\pm}]$, that is $\mathcal{A}'$ is a localization of $\mathcal{A}$ in $x_1^{(s)} \dots x_j^{(s)}$. In this case ~~$\mathcal{A}$~~ $\operatorname{Spec} \mathcal{A}'$ = principal open subset of $\operatorname{Spec} \mathcal{A} = X$, given by $x_1^{(s)} \dots x_j^{(s)} \neq 0$.

$\operatorname{Spec} \mathcal{A}' = X'$

Cluster chart in sense of Muller.

② Acyclic cluster algebras.

Def $\mathcal{A}$ is acyclic if in some seed the quiver Q has no directed cycle of mutable vertices.

Warning: It might be that after mutation such a cycle appears.

Thm (Berenstein, Fomin, Zelevinsky) If $\mathcal{A}$ is acyclic then

- $\mathcal{A}$ is finitely generated and a complete intersection
- $\mathcal{A} = \mathcal{U}$.

③ locally acyclic cluster algebras

This is the main definition for this lecture.

Def Suppose $\mathcal{A}$ is a cluster algebra, $X = \operatorname{Spec} \mathcal{A}$.
$\mathcal{A}$ is called locally acyclic if X has a finite cover by acyclic cluster charts $\{m_{s_i} \neq 0\}$.
Here m_{s_i} is a cluster monomial in some cluster s_i.

Thm (Muller) If $\mathcal{A}$ is locally acyclic then

- $\mathcal{A}$ is finitely generated, integrally closed and locally a complete intersection
- $\mathcal{A} = \mathcal{U}$
- If $\mathcal{A}$ has full rank* then X is smooth

~~then~~

* Full rank: recall that $b_{ij} = (\#\text{ arrows } i \to j) - (\#\text{ arrows } j \to i)$

$\tilde{B} = (b_{ij})$ = exchange matrix of size $(m \times (m+f))$
where there are m mutable and f frozen.

<u>Def</u> A is full rank if $\operatorname{rank} \tilde{B} = m$ in some cluster.

<u>Lemma</u> $\operatorname{rank} \tilde{B}$ does not depend on seed, invariant under mutation. Also, if A is full rank then all cluster localization of A are full rank.

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④ ~~Constructing~~ proving local acyclicity

So, locally acyclic algebras are nice, but how to find them?

<u>Def</u> A separating edge in Q is an edge $\overset{x_1}{\bullet} \to \overset{x_2}{\bullet}$
such that
- x_1, x_2 mutable
- there is at least one arrow $x_1 \to x_2$
- this arrow is not a part of any bi-infinite path through mutable

<u>Lemma</u> Suppose $x_1 \to x_2$ is a separating edge.
Then $X = \{x_1 \neq 0\} \cup \{x_2 \neq 0\}$.

<u>Pf</u> Assume for contradiction that $x_1 = x_2 = 0$ at some point p_0.
We have mutation rule $\underbrace{x_1 x_1'}_{\substack{\shortparallel \\ 0 \\ \text{at } p_0}} = \prod_{j \to 1} x_j + \underbrace{\prod_{1 \to j} x_j}_{\text{contain } x_2 \Rightarrow 0 \text{ at } p_0}$

Therefore there exists a vertex $j \to 1$ such that $x_j = 0$ at p_0.
Repeat this for $x_j x_j' = \ldots$, we get $- j_1 \to j \to 1 \to 2 \to \ldots$
and $x_2 x_2' = \ldots$ a bi-infinite path.

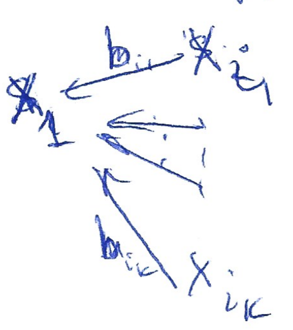
Since all frozen $\neq 0$ at p_0, all these vertices are mutable, so we get a contradiction with $1 \to 2$ being a separating edge.

<u>Corollary</u>: Assume $x_1 \to x_2$ is a separating edge and both

$$\left.\begin{array}{l} \boxed{x_1} \longrightarrow x_2 \ \cdots = \text{freeze}(x_1) \\ x_1 \longrightarrow \boxed{x_2} \ \cdots = \text{freeze}(x_2) \end{array}\right\} \text{locally acyclic}$$

Then $\mathcal{A}$ is locally acyclic.

<u>Lemma</u> Suppose x_1 is a <u>sink</u> of Q



Then $X = \{x_1 \neq 0\} \cup \{x_{i_1} \ldots x_{i_k} \neq 0\}$

<u>Pf</u>: We have mutation rule

$$x_1 x_1' = 1 + x_{i_1}^{b_{i_1}} \cdots x_{i_k}^{b_{i_k}}$$

Suppose that at some point p_0 we have $x_1(p_0) = 0$ and $x_{i_s}(p_0) = 0$ for $1 \leq s \leq k$.

Then $\left.\begin{array}{l} x_1 x_1' = 0 \text{ at } p_0 \\ x_{i_1}^{b_{i_1}} \cdots x_{i_k}^{b_{i_k}} = 0 \text{ at } p_0 \end{array}\right. \Rightarrow \boxed{1 = 0}$. Contradiction.