Tuesday, lecture 2 || Demazure product and weaves

① Demazure product

Recall that Br_n^+ is the positive braid monoid with generators $\sigma_1, \ldots, \sigma_{n-1}$. We define the Demazure product $\delta : Br_n^+ \to S_n$ in two ways:

<u>Def 1</u>: $\delta(\beta)$ is defined inductively by

- $\delta(\emptyset) = 1$
- $\delta(\beta s_i) = \begin{cases} \delta(\beta) s_i & \text{if } \delta(\beta) s_i > \delta(\beta) \\ \delta(\beta), & \text{otherwise} \end{cases}$

<u>Def 2</u> $\delta(\beta)$ is defined by the following rules:

(a) $\delta(u) = u$ if u is <u>reduced</u>

(b) $\delta(\ldots \sigma_i \sigma_{i+1} \sigma_i \ldots) = \delta(\ldots \sigma_{i+1} \sigma_i \sigma_{i+1} \ldots)$

(b') $\delta(\ldots \sigma_i \sigma_j \ldots) = \delta(\ldots \sigma_j \sigma_i \ldots)$, $|i-j|>1$

(c) $\delta(\ldots \sigma_i \sigma_i \ldots) = \delta(\ldots \sigma_i \ldots)$.

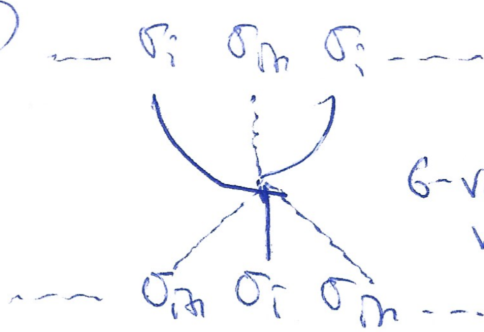
<u>Exercise</u> Prove that Def 1 and 2 are equivalent.

② Demazure weaves (= weaves)

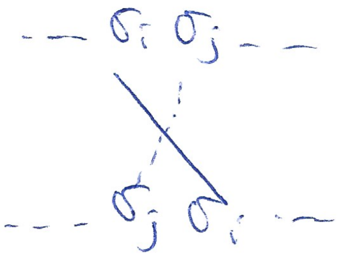
We would like to develop graphical calculus for Def 2.

Choose a color for each σ_i, then braid word = sequence of colors.

Steps (b) and (c) above $\Rightarrow$ graph relating different braid words.

(b) 

6-valent vertex

(b') 

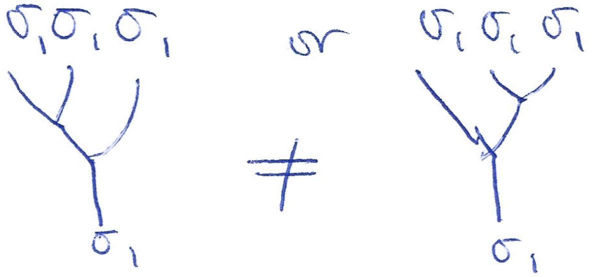
4-valent vertex $|i-j|>1$

(c) 

3-valent vertex.

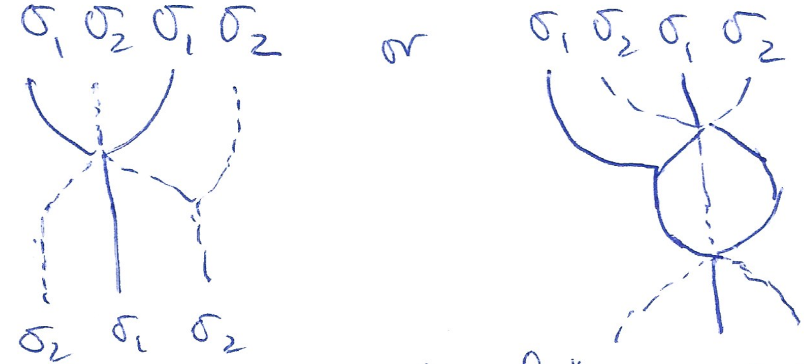
<u>Def</u> Demazure weave for β = any graph built from such pieces with β on top and $\delta(\beta)$ on bottom.

<u>Ex 1</u> $\beta = \sigma_1 \sigma_1 \sigma_1$ or $\sigma_1 \sigma_1 \sigma_1$



we do distinguish these ~~the~~ ("weave mutation")

<u>Ex 2</u> $\beta = \sigma_1 \sigma_2 \sigma_1 \sigma_2$ or $\sigma_1 \sigma_2 \sigma_1 \sigma_2$



~~Weave~~ Different weaves but we declare these equivalent.

More precisely, we have the following definition

<u>Def</u> Two weaves $\mathfrak{W}$ and $\mathfrak{W}'$ are equivalent, ~~[illegible]~~ if they have same braid words as inputs and outputs, and

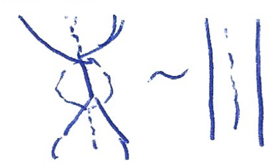
are related by the sequence of following local moves

(1) Any two weaves consisting only of 4-valent and 6-valent vertices are equivalent.

(2) Weaves in example 2 are equivalent (for $\sigma_1\sigma_2\sigma_1\sigma_2 \to \sigma_2\sigma_1\sigma_2$ or $i, i+1$)

(3) the following are equivalent if $|i-j|>1$

$\sigma_i \sigma_j \sigma_i$ $\sigma_i \sigma_j \sigma_i$

__Rmk__ For experts, (1) includes both silly things like 

and Zamolodchikov relation.

__Thm__ [CGGS1] Any two weaves from β to $\delta(\beta)$ are related by a sequence of equivalence moves and weave mutations in Ex 1.

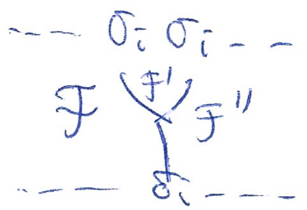
---

④ Weaves as charts

__Lemma__ Given a weave $\mathfrak{W}$ from β to β', we can associate to it a map $X(\beta') \times (\mathbb{C}^*)^{tmv} \longrightarrow X(\beta)$

__Pf__ Need to check this locally. 6-valent and 4-valent are braid moves which yield isomorphisms of braid varieties.

Trivalent



$$\mathcal{F} \xrightarrow{s_i} \mathcal{F}' \xrightarrow{\sigma_i} \mathcal{F}''$$

If $\mathcal{F} \neq \mathcal{F}''$ then there are $(\mathbb{P}^1 \setminus 2\,\text{pts}) = \mathbb{C}^*$ choices for $\mathcal{F}'$

So $X(\ldots\sigma_i\ldots) \times \mathbb{C}^* \hookrightarrow X(\ldots\sigma_i\sigma_i\ldots)$

<u>Observe</u>: Can put flags in regions of weave, $\mathcal{F} \,\big|_{\sigma_i}\, \mathcal{F}'$

means $\mathcal{F} \xrightarrow{\sigma_i} \mathcal{F}'$!

<u>Exercise</u>: work out trivalent vertex via braid matrices

<u>Cor</u> Assume $\delta(\beta) = w_0$, then all weaves from β to $\delta(\beta) = w_0$ correspond to ~~weaves~~ open embedding $X(w_0) \times (\mathbb{C}^*)^{\#\text{triv}} = (\mathbb{C}^*)^{\#\text{triv}} \hookrightarrow X(\beta)$.

This is an open torus in $X(\beta)$ and later we will prove that all such tori are cluster tori!

<u>Thm</u> Equivalent weaves yield the <u>same</u> open torus in $X(\beta)$.