Monday, Lecture 4: Finding cluster algebra structures on commutative algebras

We have seen the definition of a cluster algebra, that is very non-explicit.
How do we verify whether a given commutative $\mathbb{C}$ algebra A admits the structure of a cluster algebra?

First, A would obviously need to be an integral domain. So we will assume that

$$A = \mathbb{C}[X]$$

for some irreducible affine variety X. We let $\mathcal{F} = \mathbb{C}(X) = \mathrm{Frac}(A)$, and n = transcendence degree of $\mathcal{F}$.

A cluster structure on $\mathbb{C}[X]$ would be the data of

- Regular functions $x_1, \dots, x_n \in \mathbb{C}[X]$, s.t. $\mathcal{F} = \mathcal{F}(x_1, \dots, x_n)$
- An n vertex ice quiver Q s.t. i is frozen iff x_i is nowhere vanishing on X (ie invertible on $\mathbb{C}[X]$)

(This data would become our initial seed, let's call it $(\mathbf{x}, Q)$)

Such that:

1) $\forall$ seed (x', Q') mutation equivalent to (X, Q), we have $x' \subseteq \mathbb{C}[X]$

2) The $\mathbb{C}[x_i^{\pm 1} \mid i \text{ frozen}]$-subalgebra of $\mathbb{C}(X)$ generated by all cluster variables in all seeds mutation eq. to (X, Q), is precisely $\mathbb{C}[X]$.

· Example Top positroid in $Gr(2,n)$ (defined in previous lecture)

Def Let $k \leq n$. The top positroid variety in $Gr(k,n)$, $\Pi^{\circ}_{k,n}$, is the set of subspaces $V \in Gr(k,n)$ admitting a matrix representative $A \in Mat(k,n)$ st. $\forall$ cyclically consecutive k-element set $I \subseteq [n]$, the minor $\Delta_I(A)$ is nonzero.

Note Elementary row operations do not affect this non-vanishing condition, so $\Pi^{\circ}_{k,n}$ is well-defined.

The variety $\Pi^{\circ}_{k,n}$ is affine: since $\Delta_{\{k-n+1, \dots, n\}}(A) \neq 0$, we may assume that

$$A = \left(\begin{matrix} * & * & \cdots & * \\ * & - & & * \\ * & - & - & * \end{matrix} \; I_k \right)$$

in which case A is unique. So $\Pi^{\circ}_{k,n} \subseteq \mathbb{C}^{k(n-k)}$ is given by non-vanishing of cyclically consecutive minors.

Let's specialize to $k=2$

Note: $z_{i,1} = P_{in}$, $z_{i,2} = -P_{i\,n-1}$ so P_{ij} generate $\mathbb{C}[\Pi^\circ_{2,n}]$

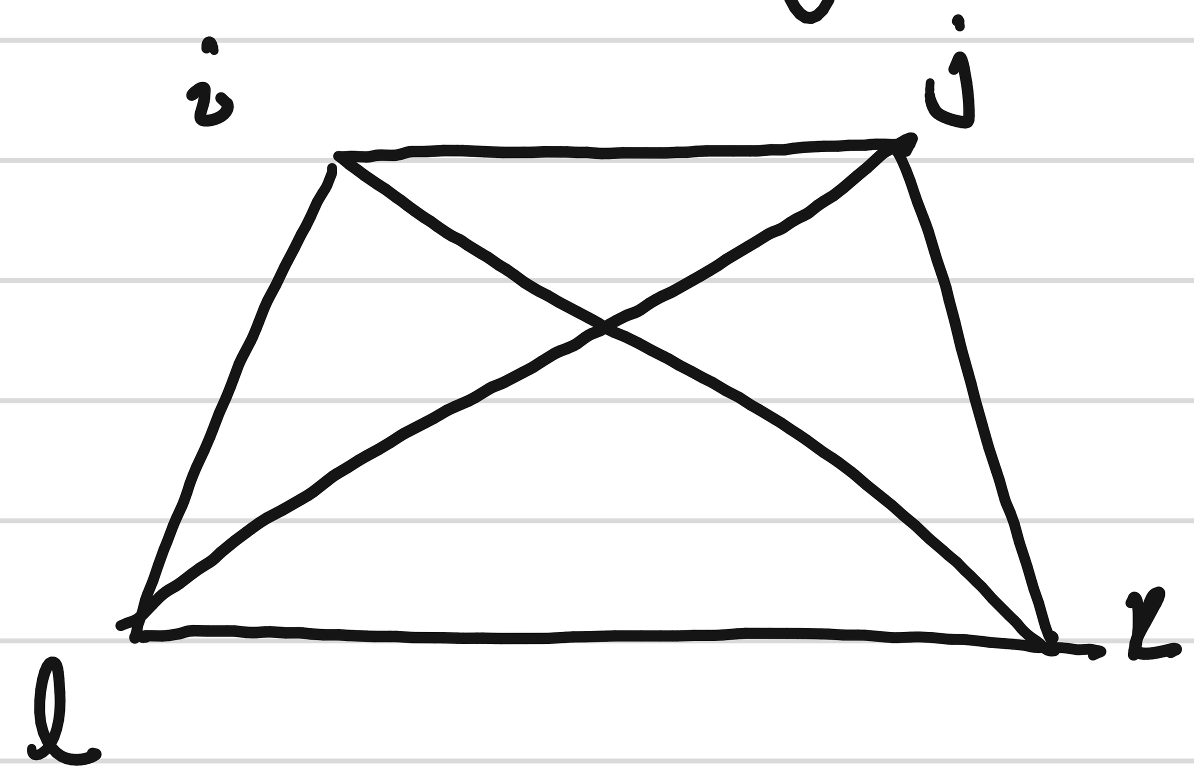
$$\mathbb{C}[\Pi^\circ_{2,n}] = \mathbb{C}\left[z_{ij} \,\middle|\, \begin{matrix} i=1,\dots,n-2 \\ j=1,2 \end{matrix}\right]\left[(z_{i,1}z_{i+1,2} - z_{i+1,1}z_{i,2})^{-1},\ z_{n-3,2}^{-1},\ z_{1,1}^{-1}\right]$$

In general, we define $P_{ij} := \det\begin{pmatrix} z_{i,1} & z_{j,1} \\ z_{i,2} & z_{j,2} \end{pmatrix}$, where we agree that

Plücker coordinates

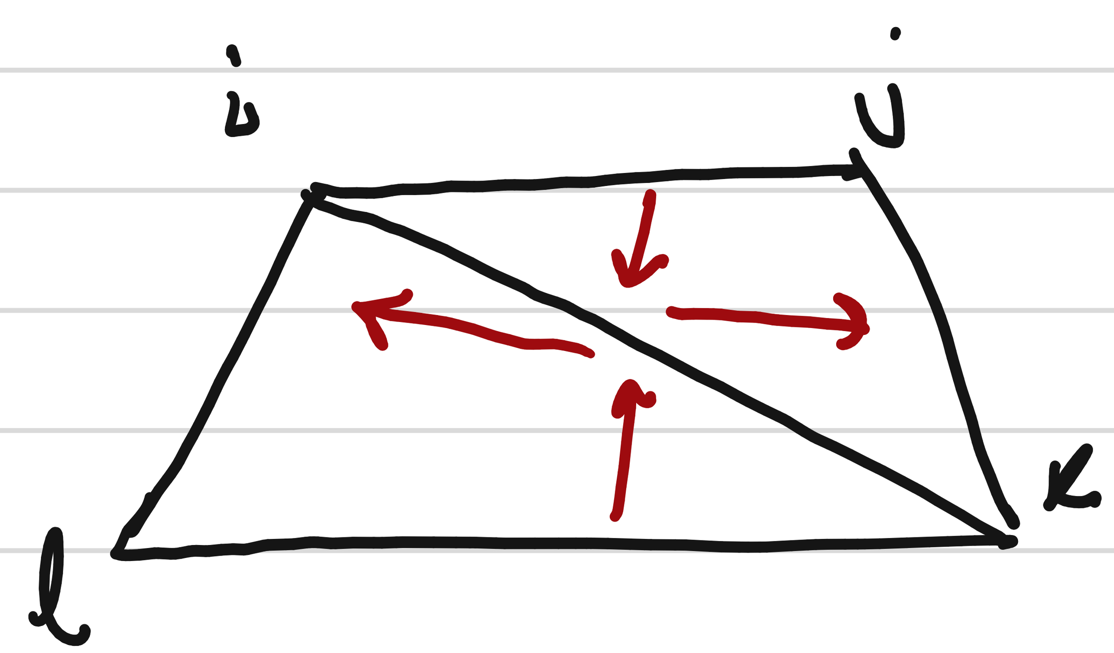
$$\begin{pmatrix} z_{n-1,1} & z_{n,1} \\ z_{n-1,2} & z_{n,2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The Plücker coordinates satisfy the Ptolemy relations



$$P_{ij}P_{kl} + P_{il}P_{jk} = P_{ik}P_{jl} \quad \underline{\text{or}}$$

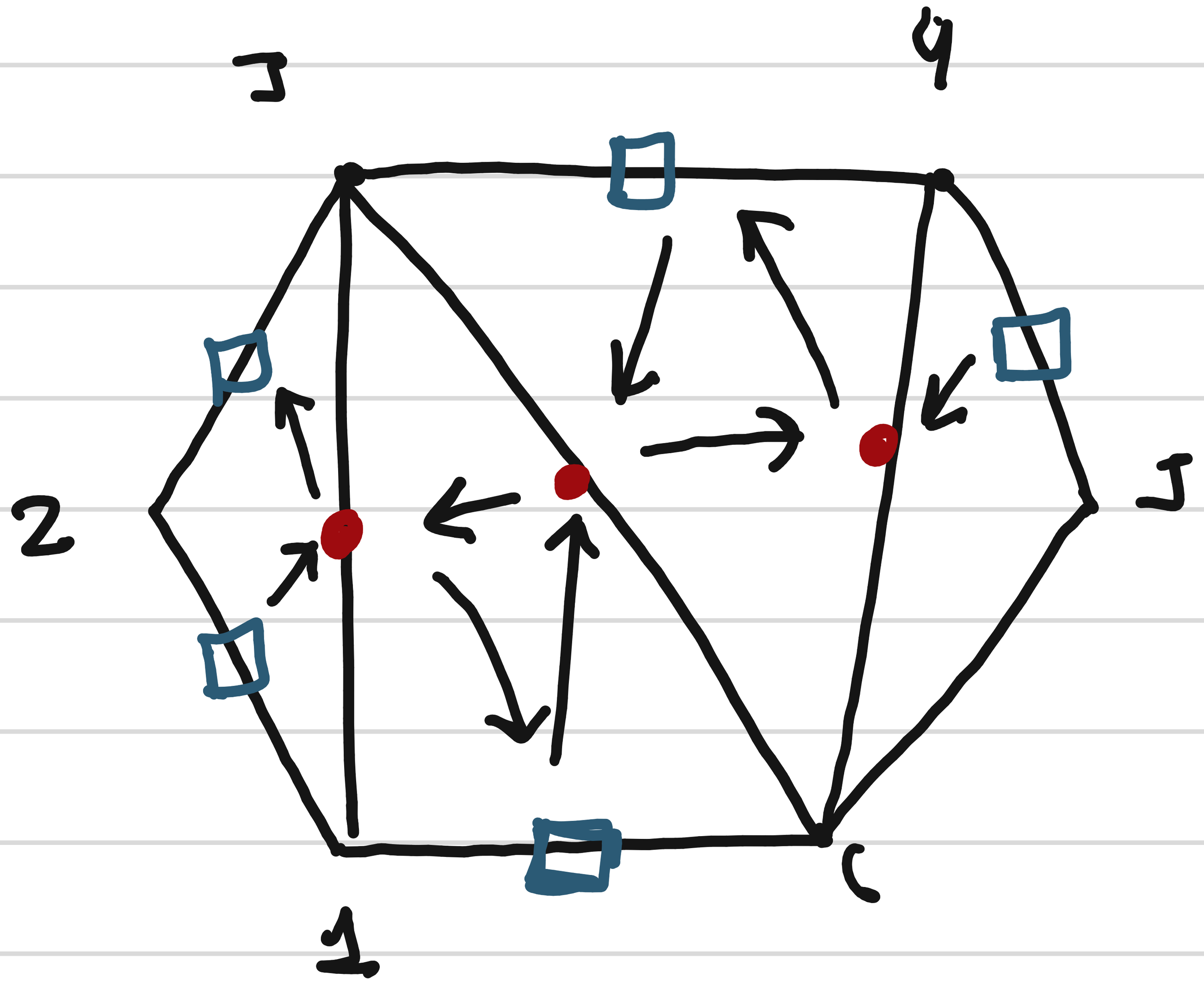
$$P_{jl} = \frac{P_{ij}P_{kl} + P_{il}P_{jk}}{P_{ik}} \quad \text{mutation!}$$



How to translate to a quiver?

So $\mathbb{C}[\Pi^{\circ}_{2,n}]$ has a cluster structure, as follows:

- Initial seed: triangulation T of an n-gon
- Mutable variables: P_{ij} where ij is a diagonal of T
- Frozen variables: $P_{i,i+1}$, i.e., these are the sides of the n-gon
- Quiver: Vertices are diagonals, arrows s.t. each triangle forms a cycle oriented counterclockwise



- All Δ's of an n-gon are related by a sequence of flips $\Rightarrow$ every Plücker belongs to some cluster $\Rightarrow \mathbb{C}[\Pi^{\circ}_{2,n}] \subseteq A(Q)$
- By mutating we never leave triangulations of n-gon $\Rightarrow A(Q) \subseteq \mathbb{C}[\Pi^{\circ}_{2,n}]$

& only finitely many seeds!

The Starfish Lemma

Let R be a finitely generated $\mathbb{C}$-algebra. Assume R is a normal domain, and let $\mathcal{F} := \mathrm{Frac}(R)$. Let $(\tilde{x}, \tilde{Q})$ be a seed in $\mathcal{F}$ s.t.

1) $\tilde{x} \subseteq R$
2) The mutable variables of $\tilde{x}$ are pairwise coprime in R
3) The frozen variables of $\tilde{x}$ are units in R.
4) $\forall$ mutable variable x_k, the mutated variable x_k' is in R and is coprime to x_k.

Then $\mathcal{U}(\tilde{x}, \tilde{Q}) \subseteq R$.

Pf sketch $R = \mathbb{C}[X]$ for a normal affine vty X. Consider

$$Y = \bigcup_{i,j} \{x_i = x_j = 0\} \cup \bigcup_k \{x_k = x_k' = 0\}$$

By the coprimeness assumptions, $\mathrm{codim}(Y) \geq 2$.

So by normality it suffices to show that any $z \in \mathcal{U}(\tilde{x}, Q) \subseteq \mathbb{C}(X)$ is regular outside of Y. Let $p \in X \setminus Y$. Then

1) If no element of $\tilde{x}$ vanishes at p, then z is regular at p b/c it's a Laurent poly in $\tilde{x}$.

2) If an element of $\tilde{x}$ vanishes at p then at most one does and it's mutable, say x_k. Then x_k' does not vanish at p (b/c $p \in X \setminus Y$) so no element of $\mu_k(\tilde{x})$ vanishes at p and we're done similarly to 1). $\blacksquare$

The starfish lemma is a bit underwhelming in that it only gives one inclusion $\mathcal{U}(\tilde{x}, Q) \subseteq R$. In fact, we'll use a refinement of it.

Suppose Q has n mutable vertices $\{1, \dots, n\}$ and m frozen ones $\{n+1, \dots, n+m\}$. We define a matrix $\tilde{B}_Q$ of size $(n+m) \times n$ (the extended exchange matrix) by

$$\tilde{b}_{ij} = \#\{\text{arrows } i \to j\} - \#\{\text{arrows } j \to i\}.$$

Note that, since Q has no oriented 2-cycles, Q determines $\widetilde{B}_Q$ and vice versa.

Example

$$Q = \begin{tikzcd} 1 \arrow[rr] & & 2 \arrow[rr] \arrow[dl] & & 3 \arrow[r] & \boxed{6} \\ & 4 \arrow[ul] \arrow[rr] & & 5 \arrow[rr] & & \boxed{7} \arrow[ul] \end{tikzcd}$$

$$\widetilde{B}_Q = \left(\begin{array}{ccccc} 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right) \text{Mutable part}$$

Def We say that Q has **really full rank** if the rows of $\widetilde{B}_Q$ generate $\mathbb{Z}^?$ as an abelian group.

The quiver in the example above has really full rank. The quiver

$$\begin{tikzcd} & & \boxed{7} \arrow[dr] & & \\ \boxed{6} \arrow[r] & 4 \arrow[ur] \arrow[d] & & 3 \arrow[ll] \arrow[r] & \boxed{8} \arrow[dl] \\ & 1 \arrow[ul] \arrow[rr] & & 2 \arrow[u] \arrow[dl] & \\ & & \boxed{5} \arrow[ul] & & \end{tikzcd}$$

does not,

nor does the quiver

$1 \rightrightarrows 2.$

Refined Starfish Lemma

Assume the conditions of the starfish lemma are satisfied and

1. Q has really full rank
2. for $\tilde{x}$ and each of its mutations, the map

$$T_{\tilde{x}} = \{p \in X \mid x_k(p) \neq 0 \ \forall x \in x\} \longrightarrow (\mathbb{C}^\times)^{n+m}$$
$$p \longmapsto (x_k(p))_{k \in Q}$$

is an iso. Then $\mathcal{U}(\tilde{x}, Q) = R$.

Pf steps: 1. Show that the full-rank condition is invariant under mutations.

2. $\forall$ mutable variable k, let P_k be the binomial appearing in the exchange reln. (so $x_k x_k' = P_k$). Show that if Q has really full rank, then the P_k are pairwise coprime. In this case we say the seed is coprime

③ (Key step) If x and y are coprime seeds related by a single mutation, then

$$\mathbb{C}[x^{\pm 1}] \cap \bigcap_{k \text{ mutable}} \mathbb{C}[\mu_k(x)^{\pm 1}] = \mathbb{C}[y] \cap \bigcap_{k \text{ mutable}} \mathbb{C}[\mu_k(y)^{-1}].$$

By these steps,

more like global sects of str. sheaf

$$\mathcal{U}(\tilde{x}, Q) = \mathbb{C}[\tilde{x}^{\pm 1}] \cap \bigcap_{k \text{ mut.}} \mathbb{C}[\mu_k(\tilde{x})^{\pm 1}] = \underbrace{\mathbb{C}[T_{\tilde{x}} \cup \bigcup_{k \text{ mut}} T_{\mu_k(\tilde{x})}]}.$$

So, if $r \in R = \mathbb{C}[X]$, r restricts to a reg function on $T_{\tilde{x}} \cup \bigcup_{k \text{ mut}} T_{\mu_k(\tilde{x})}$ and thus $r \in \mathcal{U}(\tilde{x}, Q)$. □

Notes ① The easiest way to check the conditions of the snake lemma is to check that R is a UFD and x_k, x_k' are pairwise distinct and irred. element of R.

② Steps ①, ② and ③ of the proof of the refined starfish lemma are in [Berenstein-Fomin-Zelevinsky, Cluster algebras III].

3 Tomorrow morning we'll see conditions guaranteeing $U(\vec{x}, Q) = A(\vec{x}, Q)$.