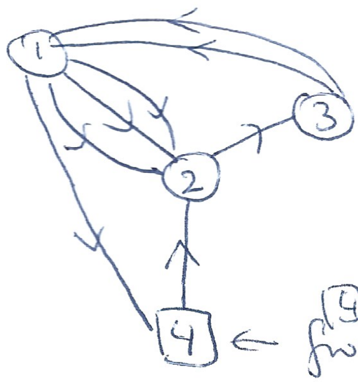
Monday, lecture 3

Cluster algebras

① A *quiver* is a directed graph Q with
- no loops (1-cycles)
- no directed 2-cycles
- Multiple edges between same vertices are allowed

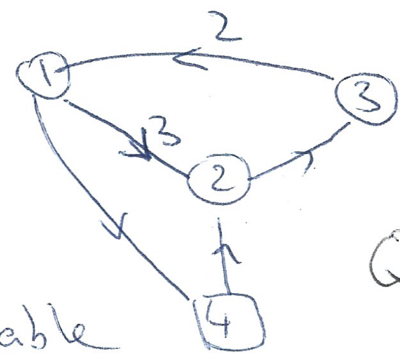
Vertices are of two types - *mutable* and *frozen*.

Ex



← ④ is frozen

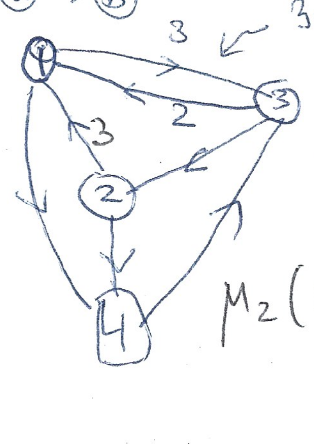
Can abbreviate to



①,②,③ are mutable

② *Quiver mutation* (at vertex i) $\mu_i(Q)$ is a new quiver obtained from Q by the following algorithm:

- For all paths ⓐ→ⓘ→ⓑ add arrow ⓐ→ⓑ
- Reverse all arrows to and from ⓘ
- Delete all directed 2-cycles

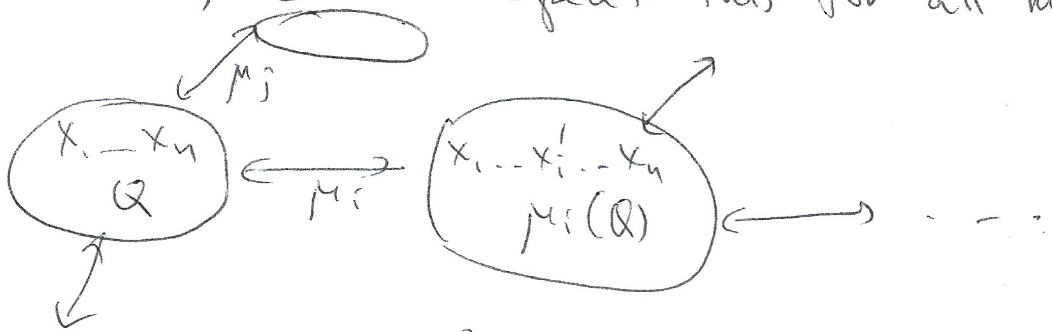


Mutations are allowed if ⓘ is mutable.

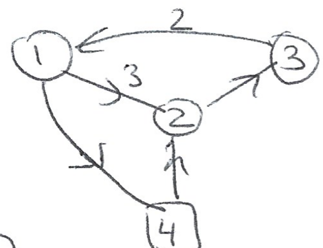
Exercise $\mu_i^2 = Id$, so mutation is an involution

③ **Cluster algebra**: start with the initial seed with algebraically independent variables $x_1 \dots x_n$ labeled by vertices of Q. We are going to inductively define many more rational functions in $\mathbb{C}(x_1 \dots x_n)$:

- Given a mutable vertex i, define $$\boxed{x_i' = \frac{\prod_{j\to i} x_j + \prod_{j \leftarrow i} x_j}{x_i} \quad (*)}$$
- We consider new seed with variables $(x_1 \dots x_i' \dots x_n)$ and the new quiver $\mu_i(Q)$.
- And so on, we can repeat this for all mutable i:

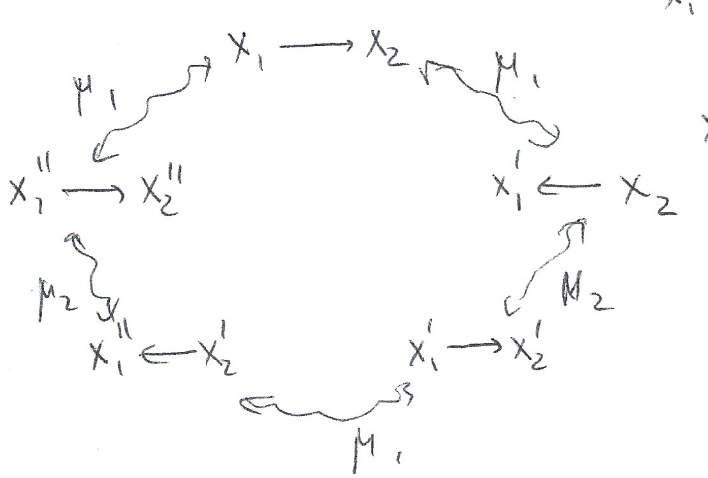


Ex For the quiver



on p.1 we get $$x_2' = \frac{x_1^3 x_4 + x_3}{x_2}$$

Ex A_2 quiver



$$x_1' = \frac{1+x_2}{x_1}, \quad x_2' = \frac{1+x_1'}{x_2} = \frac{1+x_1+x_2}{x_1 x_2}$$

$$x_1'' = \frac{1+x_2'}{x_1'} = \frac{(1+x_1+x_2+x_1x_2)/x_1x_2}{(1+x_2)/x_1} \overset{!}{=} \frac{1+x_1}{x_2}$$

$$x_2'' = \frac{1+x_1''}{x_2'} = \frac{(1+x_1+x_2)/x_2}{(1+x_1+x_2)/x_1x_2} = x_1$$

Def The cluster algebra $\mathcal{A}_Q$ is the subalgebra of $\mathbb{C}(x_1,\dots,x_n)$ generated by all cluster variables in all seeds, and by the inverses of all frozen variables.

Warning At this point, we have infinitely many generators, so $\mathcal{A}_Q$ is quite crazy from the commutative algebra perspective. Later we show that many (but not all) cluster algebras are finitely generated.

Ex (Muller) A cluster algebra of rank 3 is either generated by 6 elements or it is non-Noetherian.

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4 Laurent phenomenon

Thm (Fomin–Zelevinsky) All cluster variables in all clusters are Laurent polynomials in the initial variables $x_1,\dots,x_n$. In other words, $\mathcal{A}_Q \subset \mathbb{C}[x_1^{\pm},\dots,x_n^{\pm}]$

Def Upper cluster algebra $\mathcal{U} = \bigcap_{\text{all seeds}} \mathbb{C}[x_1^{(s)\pm},\dots,x_n^{(s)\pm}]$ where $x_i^{(s)}$ are cluster variables in the seed s.

Laurent phenomenon $\Rightarrow$ $\boxed{\mathcal{A}\subset\mathcal{U}}$. In exercise session we will discuss an example where $\mathcal{A}\neq\mathcal{U}$.

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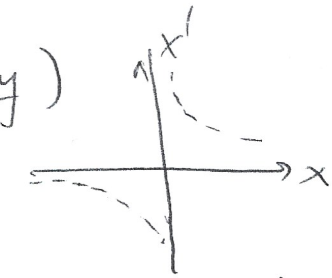
5 Cluster varieties

Given a cluster algebra $\mathcal{A}$, we define the affine variety $X=\mathrm{Spec}(\mathcal{A})$

Ex ① $\longrightarrow$ $\boxed{2}$ A, with one frozen

In exercise, we will see $A = \mathbb{C}[x, x', y^{\pm}]/(xx' = 1+y)$

$$\operatorname{Spec} A = \{xx' \neq 1\} \subset \mathbb{C}^2$$



Lemma Each seed s define a torus $(\mathbb{C}^*)^n \xhookrightarrow{i_s} X = \operatorname{Spec} A$

It is open and dense in X, and defined by non-vanishing $\{x_1^{(s)} \neq 0, \dots x_n^{(s)} \neq 0\}$.

Corollary X is irreducible.

Proof By Laurent phenomenon we have

$$\mathbb{C}[x_1^{(s)}, \dots x_n^{(s)}] \subset A \subset \mathbb{C}[x_1^{(s)\pm}, \dots x_n^{(s)\pm}]$$

If we localize all these in $x_1^{(s)} \dots x_n^{(s)}$, we get equality

$$A[(x_1^{(s)} \dots x_n^{(s)})^{-1}] = \mathbb{C}[x_1^{(s)\pm}, \dots x_n^{(s)\pm}]$$

So the ~~open~~ principal open subset in $X = \operatorname{Spec} A$ defined by $\{x_1^{(s)} \dots x_n^{(s)} \neq 0\}$ is isomorphic to $\operatorname{Spec} A[(x_1^{(s)} \dots x_n^{(s)})^{-1}] = (\mathbb{C}^*)^n$.

It is Zariski dense in X since $A \hookrightarrow \mathbb{C}[x_1^{(s)\pm} \dots x_n^{(s)\pm}]$ is injective, so the corresponding map $(\mathbb{C}^*)^n \longrightarrow \operatorname{Spec} A$ is dominant. ∎

Ex In ① $\longrightarrow$ $\boxed{2}$ we get two tori

- $\{x \neq 0, y \neq 0\} = \{x \neq 0, xx' \neq 1\}$
- $\{x' \neq 0, y \neq 0\} = \{x' \neq 0, xx' \neq 1\}$

Both are open and dense in X, but different. Also, their union does not cover the origin $(x, x') = (0, 0)$.