Braid varieties redux

<u>Recall:</u> $\beta = (i_1, \dots, i_r)$ word in $[n-1]$. The braid variety

$$X(\beta) := \{ F_\bullet^{std} = F_\bullet^0 \xrightarrow{s_{i_1}} F_\bullet^1 \xrightarrow{s_{i_2}} \cdots \longrightarrow F_\bullet^{r-1} \xrightarrow{s_{i_r}} F_\bullet^r = F_\bullet^{ast} \} \subseteq Fl_n^{\,r+1}$$

<u>Lem:</u> Fix $F_\bullet = F_\bullet(m)$. Then $\{F_\bullet' : F_\bullet \xrightarrow{s_i} F_\bullet'\} = \{F_\bullet(m B_i(z)) : z \in \mathbb{C}\} \simeq \mathbb{C}$

$$\begin{array}{cc} & i \\ i & \begin{bmatrix} \ddots & & \\ & \boxed{\begin{matrix} z & -1 \\ 1 & 0 \end{matrix}} & \\ & & \ddots \end{bmatrix} \end{array}$$

<u>Pf:</u> You showed this in exercise session.

If needed: $mB_i(z)$ has i^{th} col $z m_i + m_{i+1}$
& $(i+1)^{th}$ col $-m_i$.

This shows $\supseteq$:
$$F_\bullet(m) \xrightarrow{s_i} F_\bullet(mB_i(z)).$$

For $\subseteq$: consider $\underset{\langle m_i \rangle \neq}{F_i'/F_{i-1}'} \subseteq F_{i+1}'/F_{i-1}' = F_{i+1}/F_{i-1} = \langle m_i, m_{i+1} \rangle$

F_i'/F_{i-1}' 1-dim'l, spanned by $a m_i + b m_{i+1}$ where $b \neq 0$
" $\quad z m_i + m_{i+1}$ $\quad z \in \mathbb{C}$.

$F_{i+1}'/F_{i-1}' = \langle z m_i + m_{i+1}, -m_i \rangle$. So can choose rep $m' = mB_i(z)$ for $F_\bullet'$.

<u>Prop:</u> $X_\beta \simeq \{(z_1, \dots, z_r) \in \mathbb{C}^r : B_{i_1}(z_1) B_{i_2}(z_2) \cdots B_{i_r}(z_r)$ has zeros above anti-diagonal$\}$

affine variety!

$(F_\bullet^{std} \to F_\bullet^1 \to \cdots \to F_\bullet^r) \longleftarrow (z_1, \dots, z_r)$

where
$$F_\bullet^j = F_\bullet(\underbrace{B_{i_1}(z_1) \cdots B_{i_j}(z_j)}_{=: M_j})$$

Pf: Well-defined bc. Lemma $\Rightarrow$ $F_\bullet(M_{j-1}) \xrightarrow{s_{i_j}} F_\bullet(M_{j-1} B_{i_j}(z_j)) = F_\bullet(M_j)$

& M_r has zeros above anti-diagonal $\Rightarrow F_\bullet(M_r) = F_\bullet^{ast}$.

Bijective: let $(F_\bullet^j)_j \in X_{\mathbf{i}}$.

$F_\bullet^0 = F_\bullet(Id)$. Lemma $\Rightarrow \exists!\ z_1 \in \mathbb{C}$ s.t. $F_\bullet^1 = F_\bullet(B_{i_1}(z_1))$

Lemma $\Rightarrow \exists!\ z_2 \in \mathbb{C}$ s.t. $F_\bullet^2 = F_\bullet(B_{i_1}(z_1) B_{i_2}(z_2))$

$\vdots$

Lemma $\Rightarrow \exists!\ z_j \in \mathbb{C}$ s.t. $F_\bullet^j = F_\bullet(B_{i_1}(z_1) \cdots B_{i_j}(z_j))$

$\Rightarrow \exists!\ (z_1, \dots, z_r)$ in RHS which maps to $(F_\bullet^j)_j$. $\blacksquare$

- We utilize map in Prop. to endow $X_{\mathbf{i}}$ w/ structure of affine variety.

$$\mathbb{C}[X_{\mathbf{i}}] \simeq \mathbb{C}[z_1, \dots, z_r] \big/ \langle \text{entries above antidiagonal in } B_{i_1}(z_1) \cdots B_{i_r}(z_r) \rangle$$

Ex: $n=2$, $X_{(1)} = \{ F_\bullet \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{s_1} F_\bullet \begin{bmatrix} z & -1 \\ 1 & 0 \end{bmatrix} = F_\bullet^{ast} \} \simeq \{ z \in \mathbb{C} : z = 0 \} = \{pt\}$

$$X_{(1,1)} = \{ F_\bullet \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{s_1} F_\bullet \begin{bmatrix} z_1 & -1 \\ 1 & 0 \end{bmatrix} \xrightarrow{s_1} F_\bullet \begin{bmatrix} z_1 z_2 - 1 & z_1 \\ z_2 & -1 \end{bmatrix} = F_\bullet^{ast} \}$$

$\wr\wr$

$$\{ (z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 - 1 = 0 \} = \{ (z_1, \tfrac{1}{z_1}) : z_1 \in \mathbb{C}^\times \} \simeq \mathbb{C}^\times$$

Exercise: Find eqns for $X_{(1,1,1)}$ & draw its real points as subset of $\mathbb{R}^2$

Fact: $\operatorname{Dim} X_{\mathbf{i}} = r - \ell(w_0) = r - \binom{n}{2}$ # entries above anti-diagonal.

Open Richardson varieties as braid varieties

Sometimes $X^\circ_{\underline{i}}$ embeds into Fl_n

Recall: For $w \in S_n$, the Schubert cell $C_w := \{F_\bullet \in Fl_n : F_\bullet^{std} \xrightarrow{w} F_\bullet\} = BwB/B$

opposite Schubert cell $C^v := \{F_\bullet \in Fl_n : F_\bullet \xrightarrow{v^{-1}w_0} F_\bullet^{ast}\}$

$= (w_0 B w_0) v B/B = B_- vB/B$ $= \{g \in GL_n : \text{lower tri.}\}$

Defn: For $v, w \in S_n$, the open Richardson variety is

$$\mathcal{R}_w^v := C^v \cap C_w = \{F_\bullet : F_\bullet^{std} \xrightarrow{w} F_\bullet \xrightarrow{v^{-1}w_0} F_\bullet^{ast}\} \subseteq Fl_n$$

WARNING: Tho C^v, C_w iso to affine space, $\mathcal{R}_w^v$ is not! $\mathcal{R}_w^v = \emptyset$ if $v \nleq w$ in Bruhat order. O/w, $\mathcal{R}_w^v$ is an affine variety of dim $\ell(w) - \ell(v)$.

Used by Deodhar to understand R-polynomials (from which you can get Kazhdan-Lusztig poly)

Prop: $v, w \in S_n$. Fix reduced words $\underline{w} = s_{i_1} \cdots s_{i_p}$ & $\underline{v^{-1}w_0} = s_{j_1} \cdots s_{j_q}$.

Then $X_{(i_1, \dots, i_p, j_1, \dots, j_q)} \xrightarrow{\sim} \mathcal{R}_w^v$ is an isomorphism.

$$(F_\bullet^0 \xrightarrow{s_{i_1}} \cdots \xrightarrow{s_{i_p}} F_\bullet^p \xrightarrow{s_{j_1}} \cdots \to F_\bullet^{p+q}) \mapsto F_\bullet^p$$

In words: If $\underline{i}$ = concatenation of 2 red. words, then info of tuple $(F_\bullet^j)$ is contained in a single flag.

Pf: Exercise 2 says: If $u < us_i$ & $F_\bullet \xrightarrow{u} F'_\bullet \xrightarrow{s_i} F''_\bullet$, then $F_\bullet \xrightarrow{us_i} F''_\bullet$.

We apply this going L to R to obtain

$$\begin{tikzcd} F^{st}_\bullet = F^0_\bullet \arrow[r, "s_{i_1}"] \arrow[rr, bend right, "s_{i_1}s_{i_2}"'] \arrow[rrr, bend right=40, "s_{i_1}s_{i_2}s_{i_3}"'] \arrow[rrrrr, bend right=50, "s_{i_1}\cdots s_{i_p} = w"'] & F^1_\bullet \arrow[r, "s_{i_2}"] & F^2_\bullet \arrow[r, "s_{i_3}"] & F^3_\bullet \arrow[r] & \cdots \arrow[r] & F^p_\bullet \end{tikzcd}$$

$\cdots$

So $F^{std}_\bullet \xrightarrow{w} F^p_\bullet \Rightarrow F^p_\bullet \in C_w$

Similarly, looking at $F^p_\bullet \xrightarrow{s_{j_1}} \cdots \xrightarrow{s_{j_q}} F^{p+q}_\bullet = F^{ast}_\bullet$, we get $F^p_\bullet \xrightarrow{v^{-1}w_0} F^{ast}_\bullet \Rightarrow F^p_\bullet \in C^v$.

So map is well-defined.

Bijective: $F_\bullet \in \mathcal{R}^v_w$. By defn, $F^{std}_\bullet \xrightarrow{w} F_\bullet \xrightarrow{v^{-1}w_0} F^{ast}_\bullet$.

Exercise 2 $\Rightarrow \exists!\, F^{p-1}_\bullet$ s.t. $F^{std}_\bullet \longrightarrow F^{p-1}_\bullet \xrightarrow{s_{i_p}} F_\bullet$. Given $F^{p-1}_\bullet$,

label is $s_{i_1}\cdots s_{i_{p-1}}$

Ex. 2 $\Rightarrow \exists!\, F^{p-2}_\bullet$ s.t. $F^{std}_\bullet \longrightarrow F^{p-2}_\bullet \xrightarrow{s_{i_{p-1}}} F^{p-1}_\bullet \xrightarrow{s_{i_p}} F_\bullet$.

label is $s_{i_1}\cdots s_{i_{p-2}}$

$\vdots$

Repeating, obtain $\exists!\, F^1_\bullet, \dots, F^{p-1}_\bullet$, $F^{p+1}_\bullet, \dots, F^{p+q-1}_\bullet$ s.t. $F^{std}_\bullet \xrightarrow{s_{i_1}} F^1_\bullet \to \cdots \to F^{p-1}_\bullet \xrightarrow{s_{i_p}} F_\bullet \xrightarrow{s_{j_1}} F^{p+1}_\bullet \to \cdots \to F^{p+q}_\bullet = F^{ast}_\bullet$.

This is the unique preimage of $F_\bullet \in \mathcal{R}^v_w$. $\blacksquare$

- $Fl_n = \bigsqcup_{v \le w} \mathcal{R}^v_w$

Open positroid varieties as braid varieties

Sometimes $X_{\square}$ embeds into $Gr_{k,n}$

<u>Defn:</u> $Gr_{k,n} := \{ V \subseteq \mathbb{C}^n : \dim V = k \} \simeq \{ \text{full rk } n \times k \text{ matrices} \} \big/ \substack{\text{invertible} \\ \text{col ops}} \simeq GL_n \Big/ \begin{array}{cc} & \overbrace{\qquad}^{k} \\ k\{ & \left[\begin{array}{cc} * & * \\ 0 & * \end{array}\right] \end{array}$

$$\operatorname{colsp} A \longleftarrow [A]$$

omit

Plücker coords: $V = \operatorname{colsp} A$ where $A = \begin{bmatrix} -a_1- \\ \vdots \\ -a_n- \end{bmatrix}$. $I = \{ i_1 < \cdots < i_k \} \subseteq [n]$. $p_I(V) := \det \begin{bmatrix} - a_{i_1} - \\ \vdots \\ - a_{i_k} - \end{bmatrix}$.

These are defined only up to common scalar, since replacing A by Ag multiplies all by $\det g$.

The map $Gr_{k,n} \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$ is the <u>Plücker embedding</u>. We always consider $Gr_{k,n}$

$$V \mapsto (p_I(V))_I$$

as proj. variety wrt Plücker embedding.

- We have a natural projection $\pi_k : Fl_n \to Gr_{k,n}$

$$F_\bullet \mapsto F_k.$$

<u>Prop:</u> If $w \in S_n$ is k-Grassmannian ($w(1) < w(2) < \cdots < w(k)$ & $w(k+1) < \cdots < w(n)$), then π_k is injective on the Schubert cell C_w.

<u>Idea:</u> Every elt $F_\bullet \in C_w$ has unique rep

$$\mathfrak{M} = \begin{array}{c} \\ w(1) \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \left[\begin{array}{cccc|cccccc} * & * & * & * & 1 & 0 & \cdots & & & \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & & \\ & * & * & * & \vdots & 1 & 0 & \cdots & & \\ & 1 & 0 & 0 & \vdots & 0 & 0 & \cdots & & \\ & & * & * & \vdots & & 1 & 0 & \cdots & \\ & & 1 & 0 & \vdots & & 0 & 0 & \cdots & \\ & 0 & & * & \vdots & & & 1 & 0 & \\ & & & * & \vdots & & & 0 & 1 & \\ & & & 1 & 1 & 0 & \cdots & \cdots & 0 & \end{array}\right]$$

$\underbrace{\qquad\qquad}_{k}$ cols here are $e_{w(j)}$

If nonzero entries of M are different, colsp(1^{st} k cols) is also different.

<u>Defn:</u> Let $w \in S_n$ be k-Grassmannian, $v \leq w$. The open positroid variety $\mathring{\Pi}_w^v \subseteq Gr_{k,n}$ is

$$\mathring{\Pi}_w^v := \pi_k(\mathcal{R}_w^v).$$

<u>Note:</u> By Prop, $\mathring{\Pi}_w^v \simeq \mathcal{R}_w^v \simeq X_{(\underbrace{i_1,\ldots,i_p}_{\text{red. exp. for } w},\underbrace{j_1,\ldots,j_q}_{\text{red. exp for } v^{-1}w_0})}$. Has dim $\ell(w)-\ell(v)$

<u>Fact:</u> $Gr_{k,n} = \bigsqcup_{\substack{v \leq w \\ w \text{ } k\text{-Grass.}}} \mathring{\Pi}_w^v$

- First studied by Lusztig, Rietsch. Knutson-Lam-Speyer connect $\mathring{\Pi}_w^v$ to work of Postnikov on <u>positroids</u> & TNN $Gr_{k,n}$.

Postnikov : <u>many</u> combinatorial index sets for positroid varieties.

- Positroid var much, much simpler than $\mathcal{R}_w^v$ or $X_{\mathbf{i}}^{\circ}$!

$\mathcal{R}_w^v$, $X_{\mathbf{i}}^{\circ}$ similarly complicated.