Goal: Give an idea of how to categorify cluster structure on Richardsons

details will be missing

special case of braid var.

- What does it mean to categorify a cluster structure? [Buan-Iyama-Reiten-Scott '09]

$\mathcal{A}$ a cluster algebra		$\mathcal{C}$ a nice category (objects, morphisms, $\oplus$ makes sense)
cluster vars x_i	$\xleftarrow{\text{cluster character map}}$	(certain) indecomposable objects U_i
seed: clusters $\{x_1, \dots, x_n\}$	$\longleftarrow$	cluster tilting objects $T = U_1 \oplus \cdots \oplus U_n$
seed: quiver Q		endomorphism quiver of T, recording irreducible morphisms $U_i \to U_j$

satisfy homological condition ($\mathrm{Ext}^1(T,T)=0$ & maxl w/ this property)

- Have cluster character (CC) map $\{\text{objects of } \mathcal{C}\} \xrightarrow{\varphi} \mathcal{A}$ satisfying
  - $\varphi(m \oplus m') = \varphi(m)\varphi(m')$
  - If U_i indecomp. summand of (reachable) cluster tilting obj. T, $\varphi(U_i)$ is a cluster var.

- Note: categorification is powerful (can check if T cluster-tilting w/out reference to any other cluster tilting object). But can be hard to use; endomorphism quiver often difficult to compute.

Recall: Richardson variety $\mathcal{R}_w^v = \{F_\bullet : F_\bullet^{std} \xrightarrow{w} F_\bullet \xrightarrow{v^{-1}w_0} F_\bullet^{ast}\}$ (nonempty iff $v \leq w$)

$$\simeq X(\underline{w}\ \underline{v^{-1}w_0}) \simeq X(\underline{w_0 v}\ \underline{w^{-1}})$$

We will categorify the cluster structure on $\mathcal{R}_w^v$. Due to [Leclerc '14, Serhiyenko-SB '24]
Possibly different categorification [Ménard, '22].

conj proved

<u>Category</u>: Subcat. $\mathcal{C}_w^\vee$ of (type A) <u>preprojective algebra modules</u>.

$$P_{n-1} = \begin{tikzcd} \bullet_1 \arrow[r, bend left, "\alpha_1"] & \bullet_2 \arrow[l, bend left, "\alpha_1^*"] \arrow[r, bend left, "\alpha_2"] & \cdots \arrow[l, bend left, "\alpha_2^*"] \arrow[r, bend left, "\alpha_{n-2}"] & n-1 \arrow[l, bend left, "\alpha_{n-2}^*"] \end{tikzcd}$$

A preprojective alg. module is an assignment of

- $\mathbb{C}$ vector space V_i @ ea. vt
- linear maps $\phi_i : V_i \to V_{i+1}$, $\phi_{i*} : V_{i+1} \to V_i$ on α_i, α_i^* s.t.

$$\sum_i \big(\underbrace{\phi_i \, \phi_{i*}}_{V_{i+1} \to V_{i+1}} - \underbrace{\phi_{i*} \, \phi_i}_{V_i \to V_i}\big) = 0$$

i.e. $\phi_{1*}\phi_1 = 0$ & $\phi_i \phi_{i*} = \phi_{(i+1)*}\phi_{i+1}$ for $1 \le i \le n-2$

$\phi_{n-1*}\phi_n = 0$

e.g. simple module S_j : $V_i = \begin{cases} 0 & i \neq j \\ \mathbb{C} & i = j \end{cases}$ & all maps are zero.

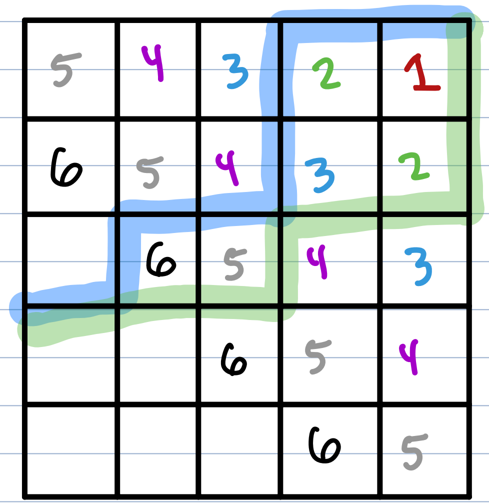
- You can imagine what $M \oplus M'$ is...
- M is <u>indecomposable</u> if can't be written as $M = M' \oplus M''$
- Morphism $\beta : M \to N$ is collection of maps $(\beta_i)_{i=1}^{n-1}$ s.t.

$$\begin{tikzcd} M & V_1 \arrow[r, bend left, "\phi_1"] \arrow[d, "\beta_1"'] & V_2 \arrow[l, bend left, "\phi_{1*}"] \arrow[r, bend left, "\phi_2"] \arrow[d, "\beta_2"] & \cdots \arrow[l, bend left, "\phi_{2*}"] \arrow[r, bend left] & V_{n-1} \arrow[l, bend left] \arrow[d, "\beta_{n-1}"] \\ N & W_1 \arrow[r, bend left, "\theta_1"] & W_2 \arrow[l, bend left, "\theta_{1*}"] \arrow[r, bend left] & \cdots \arrow[l, bend left] \arrow[r, bend left] & W_{n-1} \arrow[l, bend left] \end{tikzcd}$$

all the squares here commute.

• A morphism is <u>irreducible</u> if it doesn't factor thru another one. Note: this depends on the subcategory you're in!

Skew-shape modules



Skew shape λ/μ $\rightsquigarrow$ module

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[d] & \mathbb{C} \arrow[d] \\ \mathbb{C} \arrow[r] & \mathbb{C} \end{tikzcd}$$

all maps shown the identity map.

$\mathbb{C} \to \mathbb{C}$

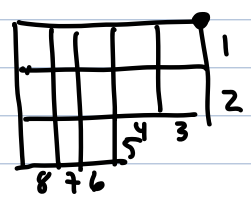
i.e.

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "{[1\ 0]}"] & \mathbb{C}^2 \arrow[l, bend left, "{\begin{bmatrix}0\\1\end{bmatrix}}"] \arrow[r, bend left, "{\begin{bmatrix}0\\1\end{bmatrix}}"] & \mathbb{C} \arrow[l, bend left, "{[1\ 0]}"] \arrow[r, bend left, "0"] & 0 \arrow[l, bend left, "0"] \arrow[r, bend left, "0"] & \mathbb{C} \arrow[l, bend left, "0"] \arrow[r, bend left, "0"] & \mathbb{C} \arrow[l, bend left, "id"] \end{tikzcd} =: M_{\lambda/\mu}$$

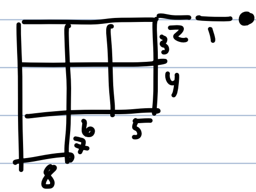
Indecomp. $\Leftrightarrow$ λ/μ connected.

Cluster-tilting objects that we construct will consist only of skew-shape modules.

Note: λ/μ is specified by 2 lattice paths, or equiv, 2 k-elt subsets which tell you where lattice paths take $\downarrow$ step.

λ 

$\leftrightarrow \{1,2,5\}$

μ 

$\leftrightarrow \{3,4,7\}$.

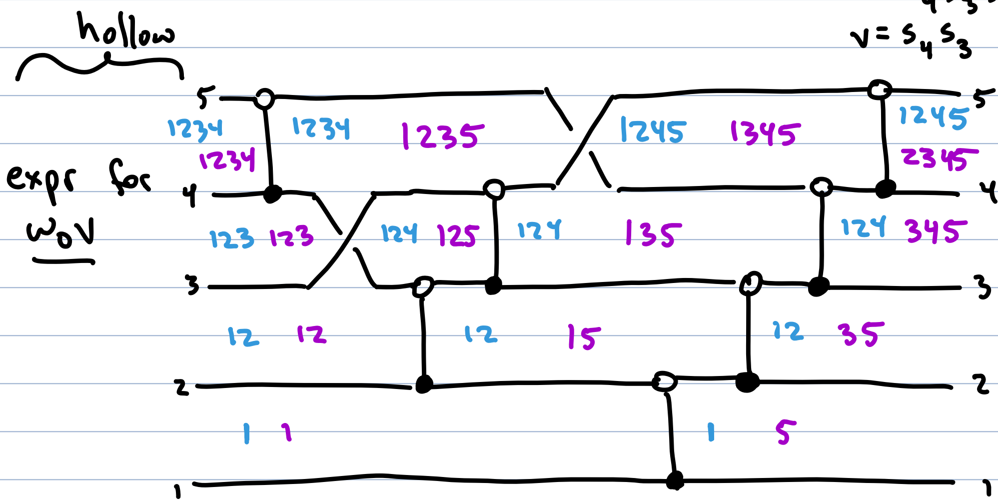
To get a cluster tilting object for $\mathcal{R}_w^v$: Draw 3D plabic gr for $\underline{w_0 v}\ \underline{w^{-1}}$

$w = s_4 s_3 s_2 s_1 s_4 s_3 s_2 s_3 s_4$

$v = s_4 s_3$

$w^{-1} = 4\,\underline{3}\,2\,3\,\underline{4}\,1\,2\,3\,4$

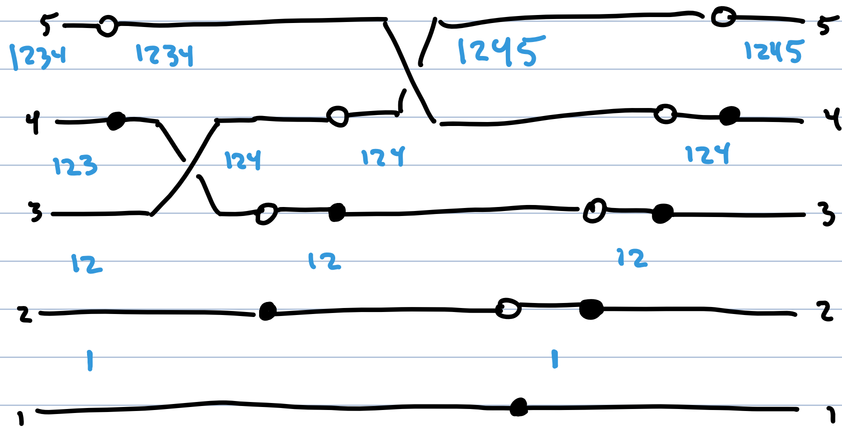
e.g.

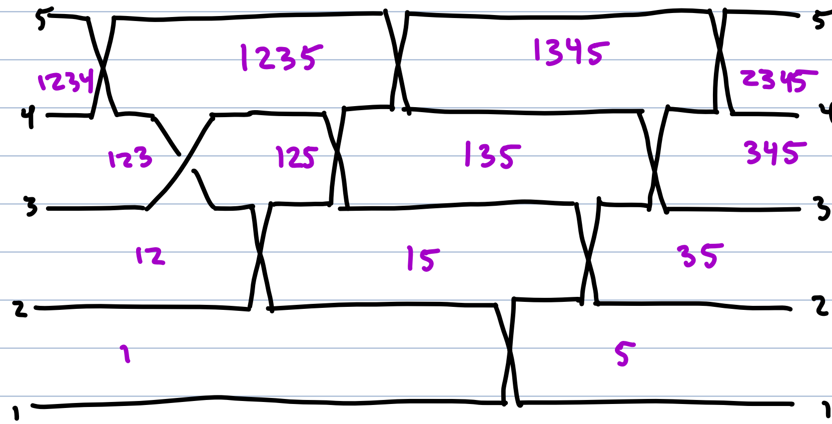


& fill ea. region in $\underline{w}^{-1}$ part by 2 subsets.

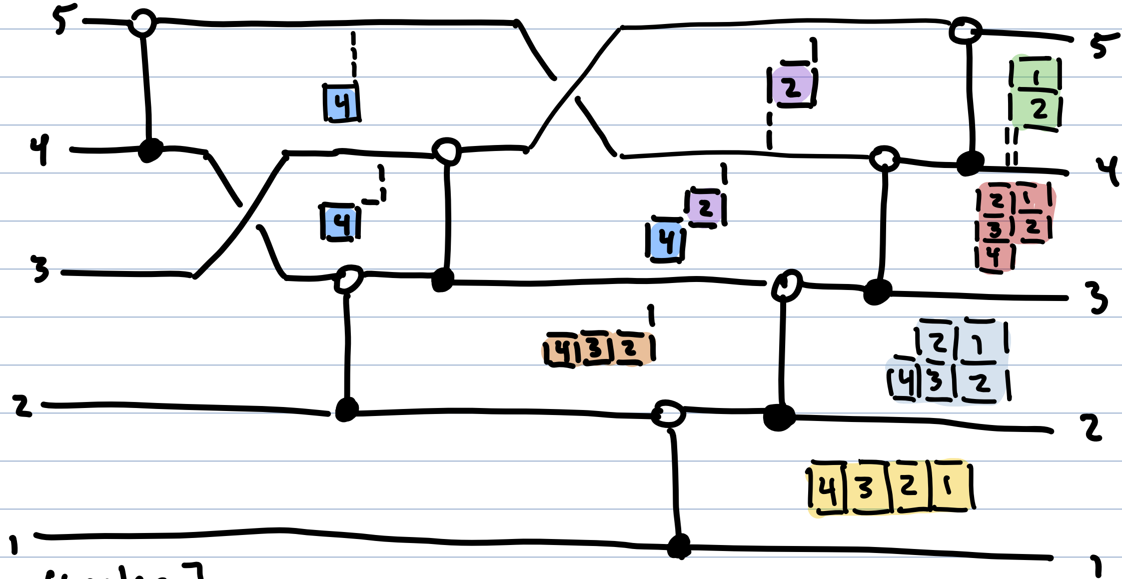
1 subset: erase bridges & record L endpt of wires below region

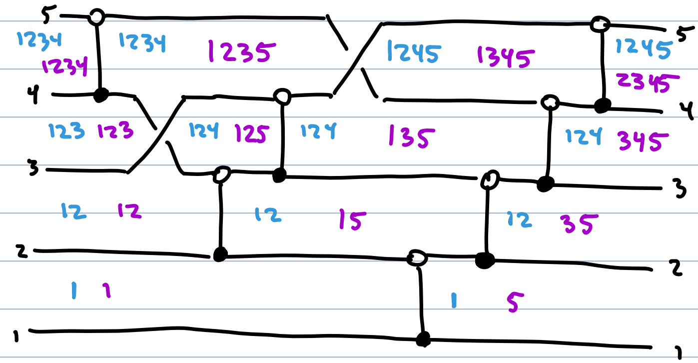
Other: turn bridges into crossings & record L endpt of wires below region.





Then interpret ea. pair of subsets as a skew-shape

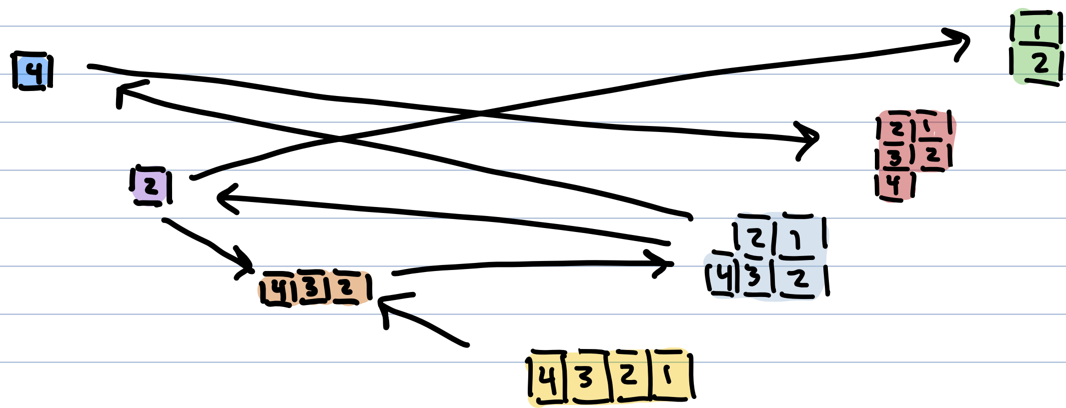




[Leclerc].

Thm: • Indecomposable summands here ↑ form a cluster tilting module T of $\mathcal{C}_w^\vee$.

[SSB] • The endomorphism quiver agrees w/ 3D plabic quiver.



[SSB] • Ψ(module in some region) = chamber minor $\circ\, \tau$

↑ specific iso $\mathcal{R}_w^\vee \longrightarrow X(\underline{w_0 v w^{-1}})$

Cor: Skew-shapes tell us factorization of chamber minors into cluster vars in Richardson case.