Comparison between cluster structures

Let $\mathbf{i}$ be a braid word.

· Melissa constructed a cluster structure on $\mathbb{C}[X(\mathbf{i})]$ using 3D plabic graphs: cluster variables are irreducible factors of chamber minors, and 3D plabic graphs help factoring the chamber minors. We'll call it the Deodhar cluster structure.

· Eugene constructed a cluster structure on $\mathbb{C}[X(\mathbf{i})]$ using weaves. For this, we need to lift to framed flags, the transition matrices are of the form $B_i(z)\chi_i(u)$ and cluster variables are monomials in the u-variables found using Lusztig cycles. We'll call this the weave cluster structure.

In this lecture, we'll show that these cluster structures coincide.

• Comparing indexing sets of cluster variables

In the Deodhar cluster structure, cluster variables are indexed by solid crossings. These are precisely the letters of $\underline{i}$ s.t. $\delta(i_1 \dots i_j) = \delta(i_1 \dots i_j i_{j+1})$.
By construction, these are in bijection with the trivalent vertices of the inductive weave $\vec{w}$.

So, in both cases, cluster variables are indexed by $\{j \mid \delta(i_1 \dots i_j) = \delta(i_1 \dots i_j i_{j+1})\} =: J$.

We'll denote x_j^D for the Deodhar cluster variables
x_j^W for the weave cluster variables (associated to the inductive weave $\vec{w}$

• Comparing cluster tori

By definition, $\bigcap_{j \in J} \{x_j^D \neq 0\} = \bigcap_{j \in J} \{x_j^W \neq 0\} =$ the Deodhar torus. Si

$$\mathbb{C}[\text{Deodhar torus}] = \mathbb{C}[(x_j^D)^{\pm 1} \mid j \in J] = \mathbb{C}[(x_j^W)^{\pm} \mid j \in J]$$

So, $\forall\, j \in J$ $\exists$ a monomial in the Deodhar cluster vars m_j^D so that

$$(\heartsuit) \quad x_j^W = m_j^D.$$

· Comparing cluster variables

We'll use the following result of Geiss-Leclerc-Schröer (in "Factorial cluster algebras")

<u>Thm</u> Let Q be an ice quiver. Assume Q doesn't have isolated mutable vertices, and let $\mathcal{A}(Q)$ be the corresponding cluster algebra. Then:

1) The units of $\mathcal{A}(Q)$ are precisely the monomials in frozen variables

2) The mutable vars are irreducible elements of $\mathcal{A}(Q)$

Assume x_j^W is mutable. By the thm above, $\exists\ j' \in J$ s.t. $x_{j'}^D$ is mutable and a monomial n_j^D in frozen variables st

$$x_j^W = x_{j'}^D n_j^D.$$

Goal $j = j'$ & $n_j = 1\ \forall j$.

- $j = j'$

Note that

$$\overline{\{(F_\bullet) \in X(\beta) \mid x_j^W(F_\bullet) = 0 \text{ and } x_i^W(F_\bullet) \neq 0\ \forall\ J \ni i < j\}} = \overline{\{(F_\bullet) \in X(\beta) \mid x_j^D(F_\bullet) = 0 \text{ and } x_i^D(F_\bullet) \neq 0\ \forall\ i < j\}}$$

$$= \{(F_\bullet) \in X(\beta) \mid F^{std} \xrightarrow{w} F_j \text{ and } w < \delta(\beta_j)\}.$$

From here, $j = j'$.

So $x_j^W = x_j^D n_j^D$. Let's eliminate denominators to have

$$n_{j,1}^D \, x_j^W = x_j^D \, n_{j,2}^D \qquad (\heartsuit\heartsuit)$$

The advantage of eliminating denominators is that this expression is valid in the polynomial algebra $\mathbb{C}[z_1, \dots, z_r]$. Adding more letters to β so that each frozen variable appearing in $(\heartsuit\heartsuit)$ becomes mutable, we conclude (by irreducibility) that

$$x_j^W = \lambda_j x_j^D \quad \text{for some } \lambda_j \in \mathbb{C}^\times.$$

If x_j^W is frozen, we may apply the same trick: add more letters so that it becomes mutable, and we get $x_j^W = \lambda_j x_j^D \quad \forall j \in J$.

Updated goal $\lambda_j = 1 \ \forall j$.

The Deodhar torus is abstractly isomorphic to a torus. We fix a multiplicative structure on it so that $\{\chi_j^W\}_{j \in J}$ is a basis of the character lattice: we can do this because $(\chi_j)_{j \in J} : T \longrightarrow (\mathbb{C}^*)^{|J|}$ is an iso.

We'll show that $\forall j \in J$, $\chi_j^D : T \longrightarrow (\mathbb{C}^*)^{|J|}$ is multiplicative, ie, a group homomorphism. If we can show this then

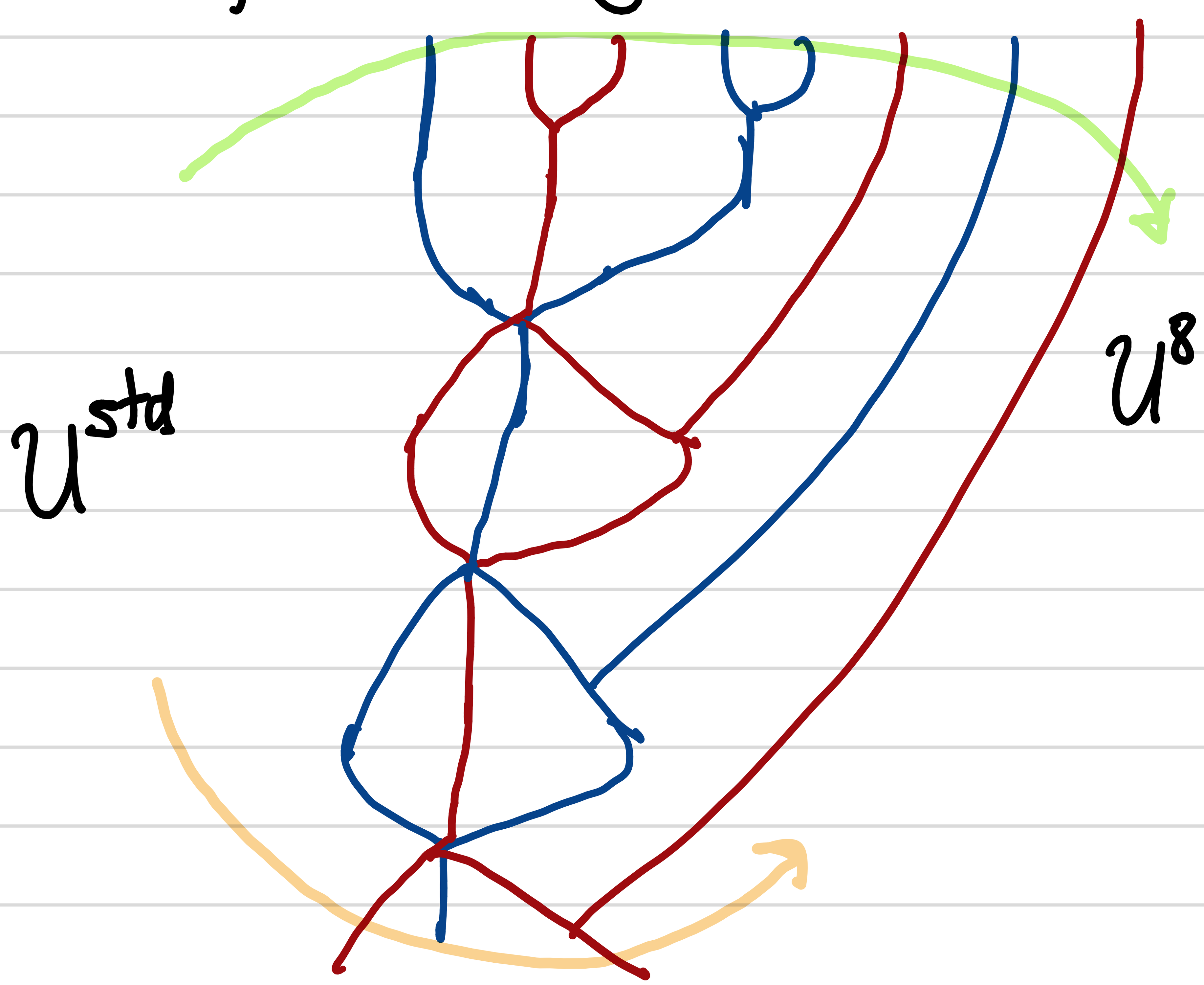
$$\lambda_j = \chi_j^W (\chi_j^D)^{-1} : T \longrightarrow (\mathbb{C}^*)^{|J|}$$

is a constant group homomorphism and thus $\lambda|_T \equiv 1$. Since T is open and $X(\beta)$ irreducible, this implies that $\lambda_j \equiv 1$.

· Since the transition matrix btw. Deodhar cluster vars and chamber minors is unitriangular, it suffices to show that the latter are group homomorphisms. Recall that

$$\Delta_j = \Delta_{\delta(\beta_j)[i_j],[i_j]} B_{\beta_j}(z_1,\dots,z_j).$$

We'll express this as a multiplicative function in the u-variables of Eugene's lectures. Since x_j^W is a monomial in these u-variables, this suffices. Let us do an example that gives the main idea. Say that $\beta_j = 1221212$



The framed flag $\mathcal{U}_8$ is

$B_{\beta_j}(z_1,\dots,z_j)\,\mathcal{U}_+$ (follow the green line)

But it is also

$B_2(\tilde{z}_1)\,\chi_2(u_1)\,B_1(\tilde{z}_2)\,\chi_1(u_2)\,B_2(\tilde{z}_3)\,\chi_2(u_3)\,\mathcal{U}_+$.

(follow the orange line) (So $\Delta_j = \Delta_{\delta(\beta_j)[i_j],[i_j]}$(orange matrix))

Now, $B_2(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & -1 \\ 0 & 1 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}}_{x_2(z)} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{s_2}$. So

$$B_2(\tilde{z}_1)\, \mathcal{X}_2(u_1)\, B_1(\tilde{z}_2)\, \mathcal{X}_1(u_2)\, B_2(\tilde{z}_3)\, \mathcal{X}_2(u_3) =$$

$$x_2(\tilde{z}_1)\, s_2\, \mathcal{X}_2(u_1) \cdot x_1(\tilde{z}_2)\, s_1\, \mathcal{X}_1(u_2)\, x_2(\tilde{z}_3)\, s_2\, \mathcal{X}_2(u_3)$$

We'd like to move all upper uni-triangular matrices x_i to the left, to have an element $g \in U_+$ so that

$$x_2(\tilde{z}_1)\, s_2\, \mathcal{X}_2(u_1) \cdot x_1(\tilde{z}_2)\, s_1\, \mathcal{X}_1(u_2)\, x_2(\tilde{z}_3)\, s_2\, \mathcal{X}_2(u_3) =$$

$$g\, s_2 s_1 s_2\, \underbrace{(s_2 s_1 \mathcal{X}_2)(u_1)\, (s_2 \mathcal{X}_1)(u_2)\, \mathcal{X}_2(u_3)}_{\text{diagonal } D}$$

By Cauchy-Binet, $\Delta_{\delta(\beta_j)[ij],[ij]}(g s_2 s_1 s_2 D) = \Delta_{[ij],[ij]}(D)$ which is multiplicative on diagonal matrices, i.e., in $\mathcal{U}$'s. To show that we can move the x matrices to the left is an exercise. This finishes the proof. ∎

We have verified cluster variables coincide. What about the quivers? This can be done using the following steps.

1) For double Bott-Samelson varieties, the quivers coincide. This follows by construction. Recall that these are words of the form $\Delta\gamma$.

2) From the quiver for $\Delta\gamma$, perform a sequence of mutations and relabelings to show that the quivers for words of the form $\gamma\Delta$ coincide. This is the key step, we need to show that we perform the same mutations and relabelings on both sides. Melissa already mentioned some of these moves on the Deodhar side yesterday.

3) The quiver for γ is obtained, on both sides, from the quiver for $\gamma\Delta$ by removing the vertices corresponding to solid crossings on the Δ side, and freezing

the remaining vertices that had an arrow to a deleted vertex.

Since the algorithm to obtain the quivers on both sides is the same, the quivers for both cluster structures coincide.