Friday, lecture 1

① In this lecture, we are going to prove that $\mathbb{C}[X(\beta)] = \mathcal{A}$ where $\mathcal{A}$ is the cluster algebra with quiver and variables constructed from weaves before. Here is a plan:

Theorem A: Given a fixed braid word β, any two seeds from weaves for β are mutation equivalent.

Theorem B: Consider braid word $\Delta\beta$ which starts from Δ (half twist). Then ~~the~~ $X(\Delta\beta) = BS(\beta)$ is the double Bott-Samelson variety discussed on Wednesday. We will prove that the Shen-Weng seed for $BS(\beta)$ corresponds to the right inductive weave for $\Delta\beta$.

Corollary $\mathbb{C}[X(\Delta\beta)] = \mathcal{A}(\Delta\beta) = \mathcal{U}(\Delta\beta)$.

Theorem C: Suppose $\beta' = \sigma_i\beta$, and we proved main thm for β'. Then $X(\beta)$ is obtained from $X(\beta')$ by cluster localization in cluster variables corresponding to the left inductive weave. If $\mathbb{C}[X(\beta')] = \mathcal{U}(\beta')$ then $\mathbb{C}[X(\beta)] = \mathcal{U}(\beta)$ (downward induction)

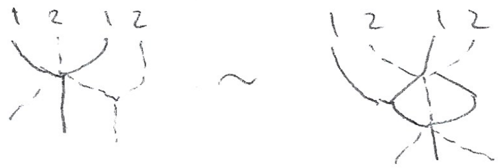
Theorem D: All z_i are cluster monomials in some clusters.

Cor $\mathbb{C}[X(\beta)] = \mathcal{A}(\beta)$.

Pf By thm D $\mathbb{C}[X(\beta)] \subset \mathcal{A}(\beta)$ and by Thm C $\mathcal{A}(\beta) \subset \mathcal{U}(\beta) = \mathbb{C}[X(\beta)]$.

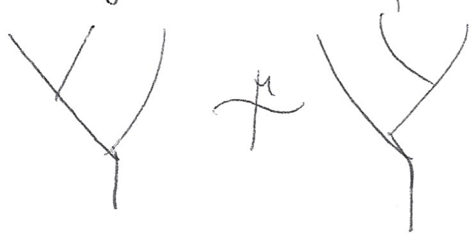
② Proof of Theorem A:

We already checked that any two quivers from weaves are mutation-equivalent. Recall that any two weaves are related by the following moves:

ⓐ 

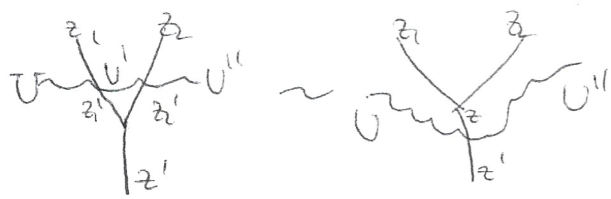
ⓑ Any two weaves with reduced (same) top & bottom and no trivalents are equivalent ("Zamolodchikov" relation)

ⓒ Changing heights of trivalents: this does not affect Lusztig cycles or quiver, but might affect yellow lines.

ⓓ 

→ exercise

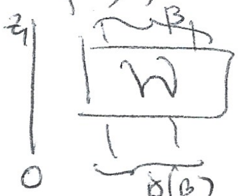
Part ⓑ is easy since no new cluster variables are created and z's and u's at the bottom are uniquely determined by the ones on the top. Parts ⓐ and ⓓ are direct checks. Part ⓒ is also a direct check, with the most subtle case



z_1', z_2' are obtained by "pushing U to the right", but crucially $\underline{z_2' = z_2}$.

③ Proof of Theorem Ⓒ

Case 1 (easy): $\delta(\sigma_i \beta) = \sigma_i \delta(\beta)$, then left inductive weaves look like this



so $z_1 = 0$

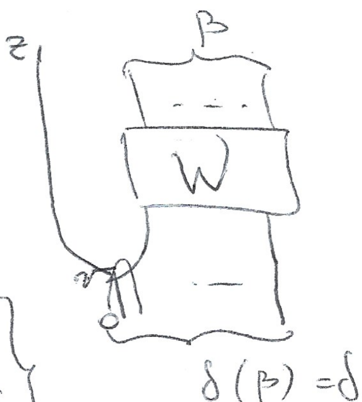
$X(\sigma_i \beta) \simeq X(\beta)$. Here we used an important

$\boxed{\text{Observation:}}$ In a Demazure weave from β to $\delta(\beta)$, all z-variables at the bottom $= 0$.

Proof: $\delta(\beta)$ is reduced, and $\mathcal{F}^{std}$ is flag on the left & $\mathcal{F}^{std}$ flag on the right $\Rightarrow$ all intermediate flags are coordinate ones

Note: u-variables at the bottom are complicated! Some monomials in frozens.

Lemma Suppose $\delta(\sigma_i \beta) = \delta(\beta) = \delta$. Then:



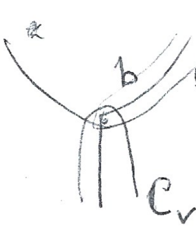
(1) the lusztig cycles and quiver agree up to the one for v. (A_v)

$\{\text{frozen for } \beta\} = \left\{\begin{array}{l}\text{frozen for } \sigma_i \beta \\ + \text{ those connected to } A_v \text{ in } Q_{\sigma_i \beta}\end{array}\right\}$

(2) Cluster vars are the same, except for A_v

(3) z_1 = cluster monomial for $\sigma_i \beta$.

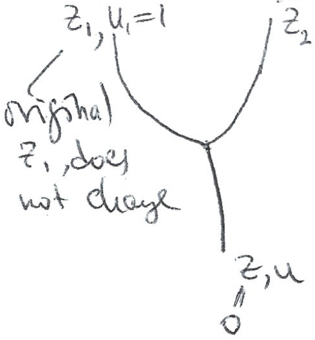
Proof: (1) By construction (2) By construction yellow lines go to the right! Do not affect σ_i



$C_i \cdot C_v = -b$

$C_i \xrightarrow{b} C_v$

Connected to C_i iff $b \neq 0$.

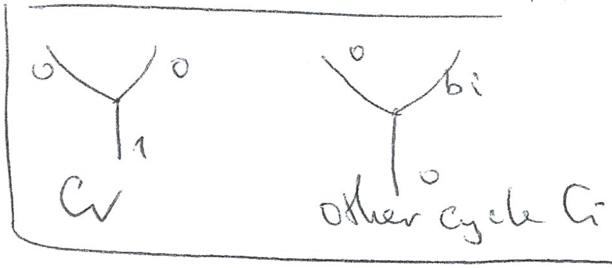


Recall: $z = z_1 - \tilde{z}_2^{-1} = 0$, so $\underline{z_2 = z_1^{-1}}$

Now $u = A_v = \tilde{z}_2 u_2 = z_1^{-1} \cdot \prod A_i^{b_i}$

$$\boxed{z_1 = A_v^{-1} \prod A_i^{b_i}}$$

cluster monomial



Also $\{z_1 \neq 0\} \simeq X(\beta) \times \mathbb{C}^*$

$X(\sigma_i \beta)$

∎

<u>Cor</u>: Assume all cluster variables for $\sigma_i\beta$ are regular, and A_v is invertible. Then all cluster vars for $X(\beta)$ are regular and those connected to A_v are invertible.

<u>Pf</u> If $\prod A_i^{b_i}$ is invertible then A_i are.

<u>Thm</u>: Assume $\mathcal{U}(\sigma_i \beta) = \mathbb{C}[X(\sigma_i \beta)]$. Then $\mathcal{U}(\beta) = \mathbb{C}[X(\beta)]$.

<u>Pf</u>: WLOG $\delta(\sigma_i\beta) = \delta(\beta) = \delta$. $X(\beta) \cong \{z_1 = 1\} \subset X(\sigma_i \beta)$

This is equivalent to $\{A_v^{-1} \prod A_i^{b_i} = 1\}$, localize A_i if $b_i > 0$

and specialize $A_v = \prod A_i^{-b_i}$.

By <u>Muller's localization lemma</u> (Tuesday, lect):

$$\mathcal{U}(\sigma_i\beta)[A_i^{\pm} : b_i > 0] = \mathcal{U}(\beta) \otimes \mathbb{C}[A_v^{\pm}]$$

$$\parallel$$

$$\mathbb{C}[X(\sigma_i\beta)][A_i^{\pm} : b_i > 0]$$

Therefore $\boxed{\mathbb{C}[X(\beta)] \subset \mathcal{U}(\beta)}$

By <u>Starfish Lemma</u> (FWZ, Cor 6.4.4) we get $\boxed{\mathcal{A}(\beta) \subset \mathcal{U}(\beta) \subset \mathbb{C}[X(\beta)]}$

Indeed, mutations for $X(\beta)$ are the same as for $X(\sigma_i \beta)$ and $X(\beta)$ is <u>smooth</u>. So $\boxed{\mathcal{U}(\beta) \simeq \mathbb{C}[X(\beta)]}$