Thursday, lecture 4 ‖ ① Framed flags

$$G = GL_n, \quad \mathbb{U} = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{matrix}\text{upper} \\ \text{unitriangular} \\ \text{matrices}\end{matrix} \subset \mathbb{B} = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

upper-triang.

$$\left\{ \begin{matrix}\text{framed} \\ \text{flags}\end{matrix} \right\} = G/\mathbb{U} \longrightarrow G/\mathbb{B} = \{\text{flags}\}$$

A <u>framed flag</u> is a flag $\{0 \subset F_1 \subset \dots \subset F_n = \mathbb{C}^n\}$ together with a choice of basis in $\mathcal{L}_i = F_i/F_{i-1}$ $(\dim \mathcal{L}_i = 1)$ (framing)

Suppose that two flags are in relative position s_i:

$$\begin{tikzcd}[column sep=small]
0 \subset F_1 \subset \dots \subset F_{i-1} \arrow[r, hook, "\mathcal{L}_i"] \arrow[dr, hook, "\mathcal{L}_i'"'] & F_i \arrow[r, hook, "\mathcal{L}_{i+1}"] & F_{i+1} \subset \dots \subset F_n = \mathbb{C}^n \\
& F_i' \arrow[ur, hook, "\mathcal{L}_{i+1}'"'] &
\end{tikzcd}$$

We have isomorphisms:

(a) $\mathcal{L}_j \cong \mathcal{L}_j'$, $j \neq i, i+1$ (b) $\mathcal{L}_i \otimes \mathcal{L}_{i+1} \cong \mathcal{L}_i' \otimes \mathcal{L}_{i+1}' \cong \Lambda^2(F_{i+1}/F_{i-1})$

(c) $\mathcal{L}_i = F_i/F_{i-1} \hookrightarrow F_{i+1}/F_{i-1} \twoheadrightarrow F_{i+1}/F_i' = \mathcal{L}_{i+1}'$ { (c) is iso since F_i is transversal to F_i'

and similarly $\mathcal{L}_i' \longrightarrow \mathcal{L}_{i+1}$.

<u>Def</u> Two <u>framed</u> flags F_{fr}, F_{fr}' ~~are~~ in <u>strong</u> relative position s_i if $\exists g \in GL_n$ such that $gF_{fr} = F_{fr}(I)$ = framed standard flag

$gF_{fr}' = F_{fr}(s_i)$

<u>Lemma/exercise</u> F_{fr}, F_{fr}' are in strong relative position s_i iff the isomorphisms (a) - (c) all agree with the framing

iff $F_{fr} = F_{fr}(M)$, $F_{fr}(M') = F_{fr}'$ and $M' = M B_i(z) U$ for $U \in \mathbb{U}$, $z \in \mathbb{C}$

<u>Rmk</u> 1) We assume $s_i = B_i(0)$.

2) Given a matrix $M \in G$, we define $F_{fr}(M) = \langle m_1 \rangle \subset \langle m_1, m_2 \rangle \subset \dots$ with framing $m_i \in F_{fr}^i(M)/F_{fr}^{i-1}(M)$

Lemma/exercise Suppose two framed flags $\mathcal{F}_{fr}$, $\mathcal{F}'_{fr}$ are such that ~~...~~ the underlying flags are in position s_i, and (a)–(b) are iso (but not necessarily (c))

Then $\mathcal{F}_{fr} = \mathcal{F}_{fr}(M)$, $\mathcal{F}'_{fr} = \mathcal{F}_{fr}(M')$ where

$M' = M B_i(z) X_i(u)$ and $X_i(u) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & u & \\ & & & u^{-1} \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$ $u \in \mathbb{C}^*$

② Framed flags for a weave

Construction: $\mathcal{W}$ = weave from β to $\delta(\beta)$

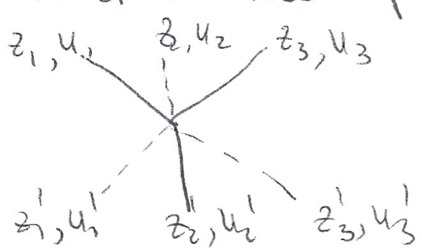
- On every edge, choose some z_e, u_e
- In every region, a framed flag
- Given two regions separated by an edge of color i:

$\mathcal{F}_{fr}(M) \Big|_{s_i} \mathcal{F}_{fr}(M')$, $M' = M B_i(z_e) X_i(u_e) U$

$U \in \mathbb{U}$

- Alternatively, a matrix in each region, but will need several matrices for the same framed flag. Separate these by "yellow line" (see below).
- Standard framed flag on the left.

Thm This assignment of framed flags exists and is unique, given z_i on the top and all u_e, provided consistency condition at each 6v vertex:



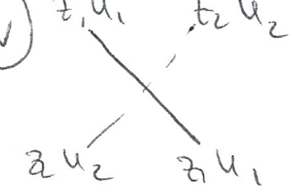
$$\boxed{u_1 u_2 = u'_2 u'_3, \quad u_2 u_3 = u'_1 u'_2}$$

Proof: We go from top to bottom and treat each ~~vertex~~ vertex separately.

(6v) $B_i(z_1) X_i(u_1) B_{i+1}(z_2) X_{i+1}(u_2) B_i(z_3) X_i(u_3) =$ _____ / _____ with primes (exercise)

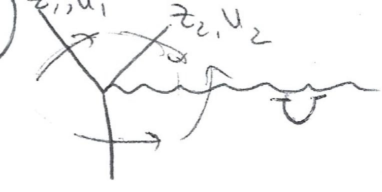
iff consistency condition holds

4v



clear

3v



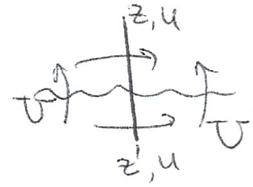
$$B_i(z_1)X_i(u_1)B_i(z_2)X_i(u_2) =$$

$$= B_i(z)X_i(u)\,U \qquad U \in \mathbb{U}$$

if $\boxed{z_2 \neq 0}$ $\boxed{z = z_1 - u_1^{-2} z_2^{-1} \qquad u = z_2 u_1 u_2}$

NB: The matrices above/below yellow line differ by U, but the corresponding flags are the same.

Propagation of yellow lines



For $U \in \mathbb{U}$, $z \in \mathbb{C}$, $u \in \mathbb{C}^*$ $\exists!\, z'$ such that $U' \in \mathbb{U}$

$$U B_i(z) X_i(u) = B_i(z') X_i(u) U'$$

So at each 3v we create a triangular matrix U, then push it all the way to the right but change z's along the way. ∎

Thm Given z_i on the top, there is a unique collection of rational function A_i, one for each 3-valent vertex, such that $\boxed{\text{Construction}}$ holds and $\boxed{u_e = \prod_i A_i^{\gamma_i(e)}}$ where $\gamma_i(e)$ = weight of e in the Lusztig cycle corresponding to the trivalent vertex i. $\boxed{\text{NB}}$: u_e at the top $= 1$.

Pf: First, let us define A_i. At each trivalent, we have

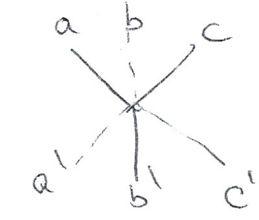
$$u_1 = \prod_i A_i^{\gamma_i(e_1)} \qquad u_2 = \prod_i A_i^{\gamma_i(e_2)}$$

$$u = A \cdot \prod_i A_i^{\gamma_i(e)}$$ where A is new (unknown) cluster variable

Since $u = z_2 u_1 u_2$, we get

$$A = z_2 \cdot \prod_i A_i^{\gamma_i(e_1) + \gamma_i(e_2) - \gamma_i(e)}$$

Second, we need to check consistency condition at (6v) for a Lusztig cycle



$a+b = b'+c'$
$b+c = a'+b'$

$\Rightarrow u_1 u_2 = u_2' u_3', \quad u_2 u_3 = u_1' u_2',$

<u>Thm</u> (Main thm of CGGLSS)

1) A_i are in fact regular functions on $X(\beta)$
2) $X(\beta)$ has a cluster structure, where each weave W defines a seed (but there're many more seeds)

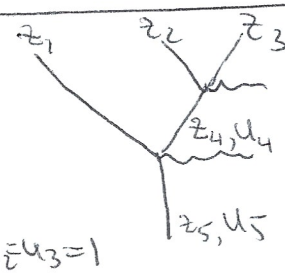
<u>Quiver</u> = intersections between Lusztig cycles

<u>Cluster variables</u> = A_i as above

3) Any two seeds for different weaves are mutation equivalent

---

<u>Ex</u>



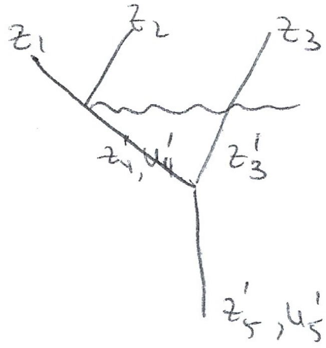
$u_1 = u_2 = u_3 = 1$

$z_4 = z_2 - z_3^{-1}, \quad u_4 = z_3$

$z_5 = z_1 - z_4^{-1}, \quad u_5 = z_4 u_4 = z_2 z_3 - 1$

$\boxed{A_1 = u_4 = z_3} \quad A_2 = u_5 = \boxed{z_2 z_3 - 1}$

<u>Ex</u>



$z_4' = z_1 - z_2^{-1} \quad u_4' = z_2$

$B_1(z_1) B_1(z_2) B_1(z_3) = B_1(z_1 - z_2^{-1}) X_1(z_2) \begin{pmatrix} 1 & -z_2^{-1} \\ 0 & 1 \end{pmatrix} B_1(z_3) =$

$= B_1(z_1 - z_2^{-1}) X_1(z_2) B_1(z_3')$

$z_3' = z_3 - z_2^{-1} \quad u_5' = z_3' u_4' = z_2 z_3 - 1$

$\boxed{A_1' = u_4' = z_2} \quad \boxed{A_2 = u_5' = z_2 z_3 - 1}$

$A_1' = \dfrac{1 + A_2}{A_1}$ $\qquad A_1 - \boxed{A_2}$