Lusztig cycles

Setup. • β = braid word, $\delta(\beta)$ = Demazure product

• $\mathfrak{W}$ = Demazure weave from β to $\delta(\beta)$.

Goal for this lecture:

① Define a collection of "Lusztig cycles" $\text{edges}(\mathfrak{W}) \to \mathbb{Z}_{\geq 0}$

$\#\text{cycles} = \#\text{trivalent vertices in } \mathfrak{W} = \ell(\beta) - \ell(\delta(\beta))$

② For two cycles C_1, C_2 define "intersection number" $C_1 \circ C_2$

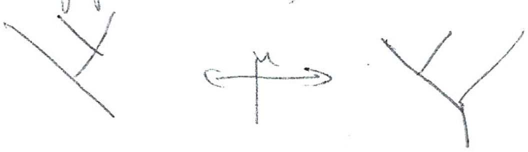
Antisymmetric $C_1 \cdot C_2 = -C_2 \circ C_1$

$\Rightarrow$ Skew symmetric matrix $B = (b_{ij}) = (C_i \circ C_j)$

$\Rightarrow$ Q quiver, b_{ij} arrows from i to j if $b_{ij} \geq 0$

③ Thm (a) Suppose that two weaves $\mathfrak{W}, \mathfrak{W}'$ are equivalent. Then $Q = Q'$ the corresponding quivers are the same

(b) Suppose $\mathfrak{W}, \mathfrak{W}'$ are related by weave mutation



Then Q, Q' are related by quiver mutation.

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① Lusztig cycles Let v be a trivalent vertex in $\mathfrak{W}$

The corresponding cycle C_v is defined as follows

$\boxed{C_v : \text{edges}(\mathfrak{W}) \to \mathbb{Z}_{\geq 0}}$

• Above v, $C_v = 0$

• At v:

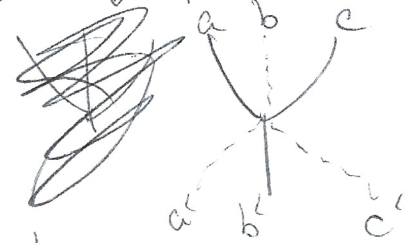
"C_v starts at v"

· Propagate down as follows:

3-valent not v

$a' = \min(a_1, a_2)$

6-valent

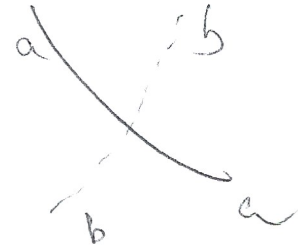


$b' = \min(a, c)$

$a' = b + c - \min(a, c)$

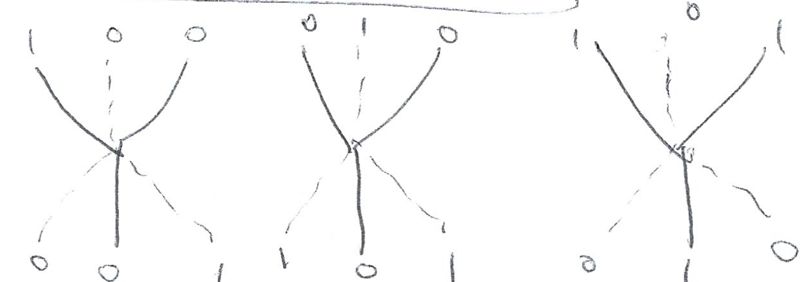
$c' = a + b - \min(a, c)$

4-valent



Sample propagations

cycle "stops"



"goes through" "split" "merge"

Big example here (draw on board in advance)

Aside | Tropicalization $x_i(t) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 \; t & \\ & & 1 & \\ & & & \ddots \; 1 \end{pmatrix} = \exp(tE_i)$

Lusztig:

* $x_i(t_1)\, x_i(t_2) = x_i(t_1 + t_2)$
* $x_i(t_1)\, x_{i+1}(t_2)\, x_i(t_3) = x_{i+1}\left(\frac{t_2 t_3}{t_1 + t_3}\right) x_i(t_1 + t_3)\, x_{i+1}\left(\frac{t_1 t_2}{t_1 + t_3}\right)$
* $x_i(t_1)\, x_j(t_2) = x_j(t_2)\, x_i(t_1) \quad |i - j| > 1$
* $u = s_{i_1} \cdots s_{i_k}$ reduced, A upper triangular

<u>If</u> A has a factorization $x_{i_1}(t_1) \cdots x_{i_k}(t_k) = A$ then t_j are uniquely determined by A

$\longrightarrow x_{i_1}(t^{a_1}) \cdots x_{i_k}(t^{a_k})$ "tropicalization"

slice of W

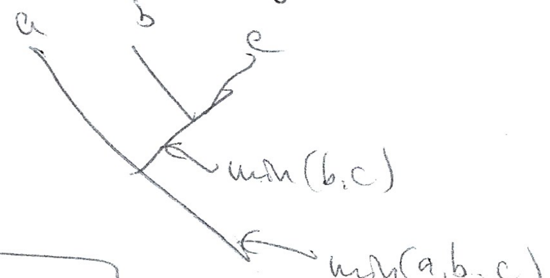
valuation v = order in t

Exercise Our propagation rules are tropicalization of Lusztig's.

Corollary Assume we have a piece of W with

Then the values of C on ~~[illegible]~~ output edges are determined by input and do not depend on intermediate weave.

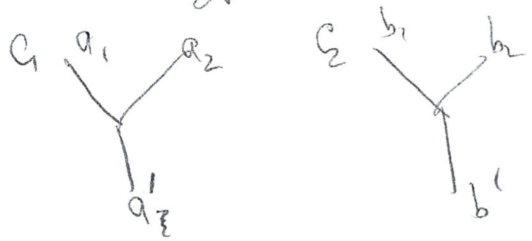
Ex



② Intersection numbers

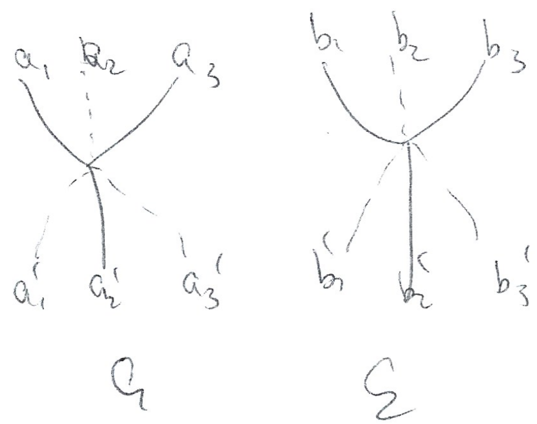
$$C_1 \circ C_2 = \sum_{3v} (C_1 \circ C_2)_{loc} + \sum_{6v} (C_1 \circ C_2)_{loc}.$$

3v



$$(C_1 \circ C_2)_{loc} = \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a' & a_2 \\ b_1 & b' & b_2 \end{vmatrix}$$

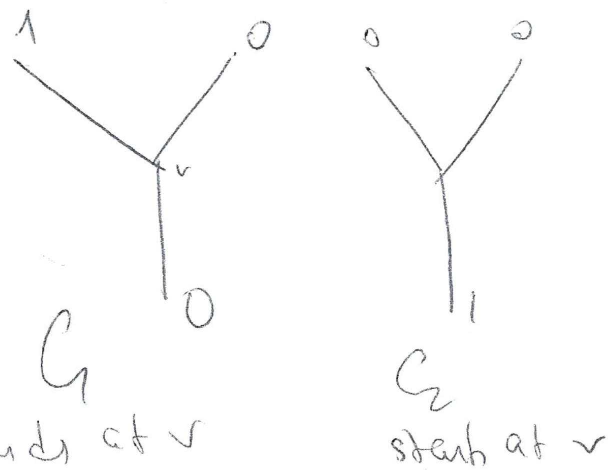
6v



$$(C_1 \circ C_2)_{loc} = \frac{1}{2}\left[\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 1 \\ a_1' & a_2' & a_3' \\ b_1' & b_2' & b_3' \end{vmatrix} \right]$$

Fact This is an integer.

Ex (a)



$$C_1 \circ C_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = +1$$

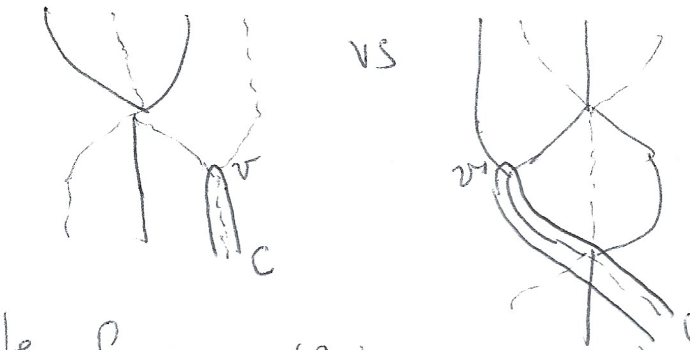
(b)

G_1 $\qquad$ G_2

$$(G_1\circ G_2)_{loc} = \frac{1}{2}\left[\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}\right]$$

$$= \frac{1}{2}\left[-1 \ -1\right] = -1$$

<u>Thm</u> (a)



* C_1, C_2 = two cycles which start above v

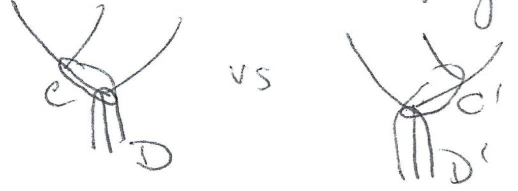
$$(C_1\circ C_2)_w = (C_1'\circ C_2')_{w'}$$

* C = cycle starting at v (resp. C') (resp. v')

$$(C_i, C)_w = (C_i', C')_w$$

<u>Proof</u>: brute force. <u>NB</u> By Corollary, cycles below this fragment agree, so this a <u>local</u> computation!

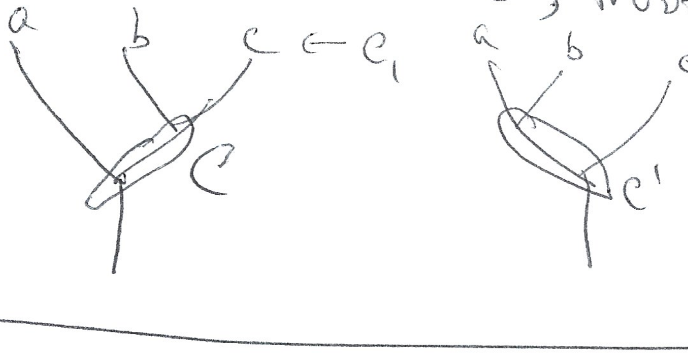
(b)



two cycles C, D (resp. C', D') starting here + C_1, C_2 starting above.

<u>Claim</u>: Q' = mutation of Q at C; Proof: brute force!

<u>Exercise</u>



Compute $C_1\circ C$, C_1', C' and prove $C_1\circ C = -C_1'\circ C'$!

(2b) Bonus: boundary intersections

$\beta = \sigma_{i_1} \ldots \sigma_{i_k}$ Define a sequence of roots

$$\rho_j = \sigma_{i_1} \ldots \sigma_{i_{j-1}}(\alpha_j)$$

Def $\beta = \sigma_{i_1} \cdots \sigma_{i_k}$ $\quad$ C has weights $c_1, \dots, c_k$

C' has weights $c'_1, \dots, c'_k$

$$\#_\beta (C \cdot C') = \frac{1}{2} \sum_{i,j=1}^{k} \operatorname{sgn}(i-j)\, c_i c'_j \,(p_i, p_j^*)$$

Lemma Suppose $\mathfrak{w}$ = weave with only 6-valent and 4-valent vert.

C, C' two cycles which start above γ

$$\gamma$$
$$\boxed{\mathfrak{w}}$$
$$\gamma'$$

Then $(C \cdot C')_{\mathfrak{w}} = \#_\gamma (C \cdot C') - \#_{\gamma'} (C \cdot C')$

Pf: local computation at 6-valent and 4-valent.

Cor If $\mathfrak{w}, \mathfrak{w}'$ are two weaves from γ to γ'

then $(C \cdot C')_{\mathfrak{w}} = (C \cdot C')_{\mathfrak{w}'}$.