Recap: $X(\beta)$ braid variety

β pos. braid word for β $\rightsquigarrow$ $T_\beta \subseteq X(\beta)$ Deodhar torus

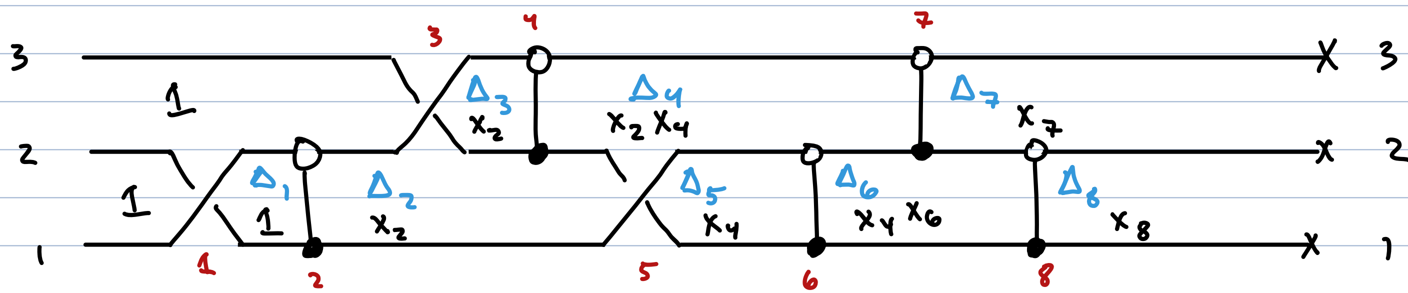
$$T_\beta = \{\underbrace{\Delta_j}_{\text{grid minors}} \neq 0 \text{ for } j \in [r]\} = \{\underbrace{\Delta_j}_{\text{chamber minors}} \neq 0 \text{ for } j \in J_\beta\}$$

(Proposed) cluster var $\{x_d\}_{d \in J_\beta}$ are irred. factors of Δ_j, which we can understand by analyzing Deodhar hypersurfaces V_d' & $\mathrm{ord}_{V_d'} \Delta_j$ = exponent of x_d in Δ_j.

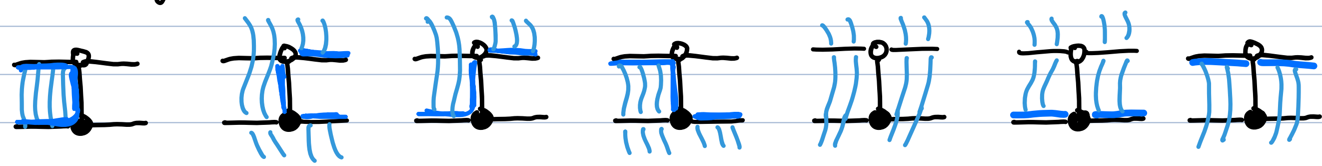
Can track factorization combinatorially using 3D plabic graphs & soap films

$\beta = (1,1,2,2,1,1,2,1) \in Br_3^+$

3D plabic graph G_β

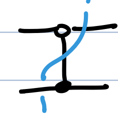


Soap film propagation: (to R)



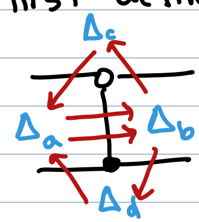
S_d can only "pass thru" bridges like this:

not like



Today: define seed using this combinatorics & start pf.

<u>Defn</u>: Given β pos. braid word, first define a quiver Q' w/ vt = grid minors & arrows given by placing this :

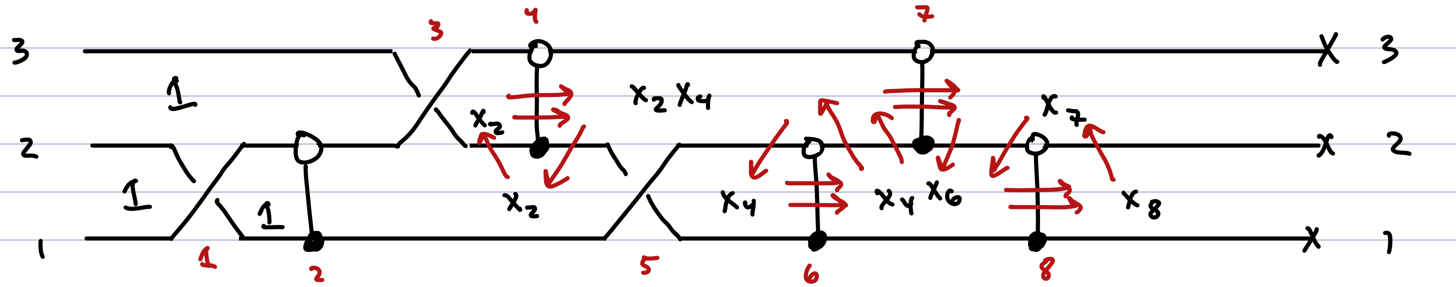


around ea. bridge

<u>Note</u>: can also place this around ea. crossing; won't change

final quiver

Q':

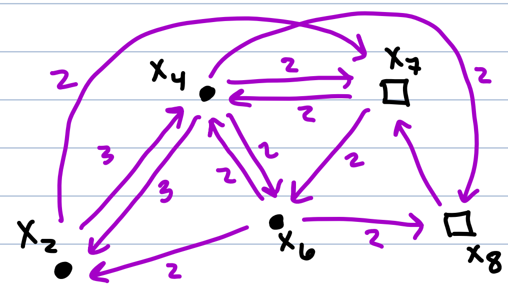


The quiver Q_β has

- vt $\{x_d\}_{d \in J_\beta}$ $\leftrightarrow$ bridges $\leftrightarrow$ soap films.
  x_d is <u>frozen</u> if x_d appears in open chamber on R.
  O/w mutable.
- arrows are given as follows: for ea. arrow $\Delta_a \longrightarrow \Delta_b$ in Q' & ea. $x_d \mid \Delta_a$, $x_{d'} \mid \Delta_b$, draw arrow

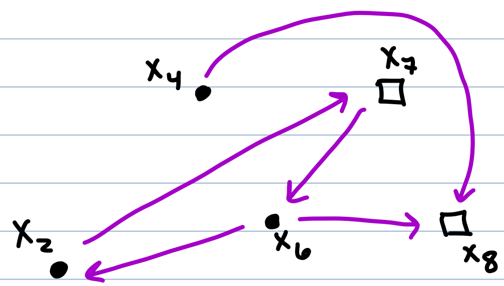
$$x_d \longrightarrow x_{d'} .$$

e.g. (not drawing loops)



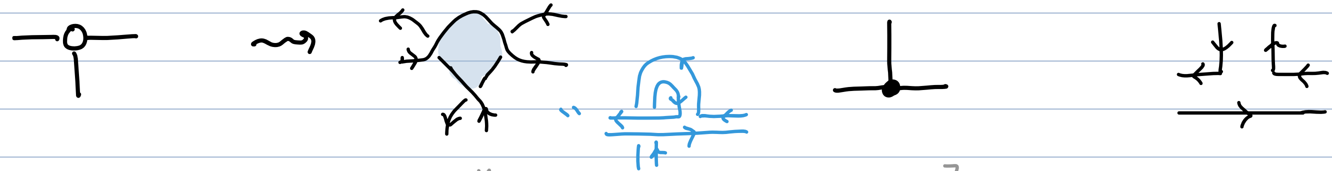
Then delete max'l collection of 2-cycles, arrows btw frozens, & divide # arrows by 2.

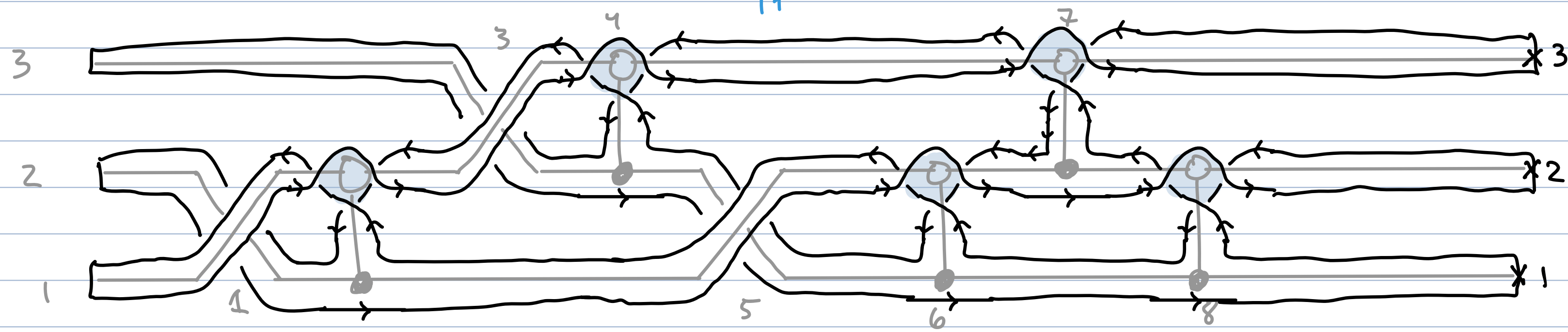
Q_β :



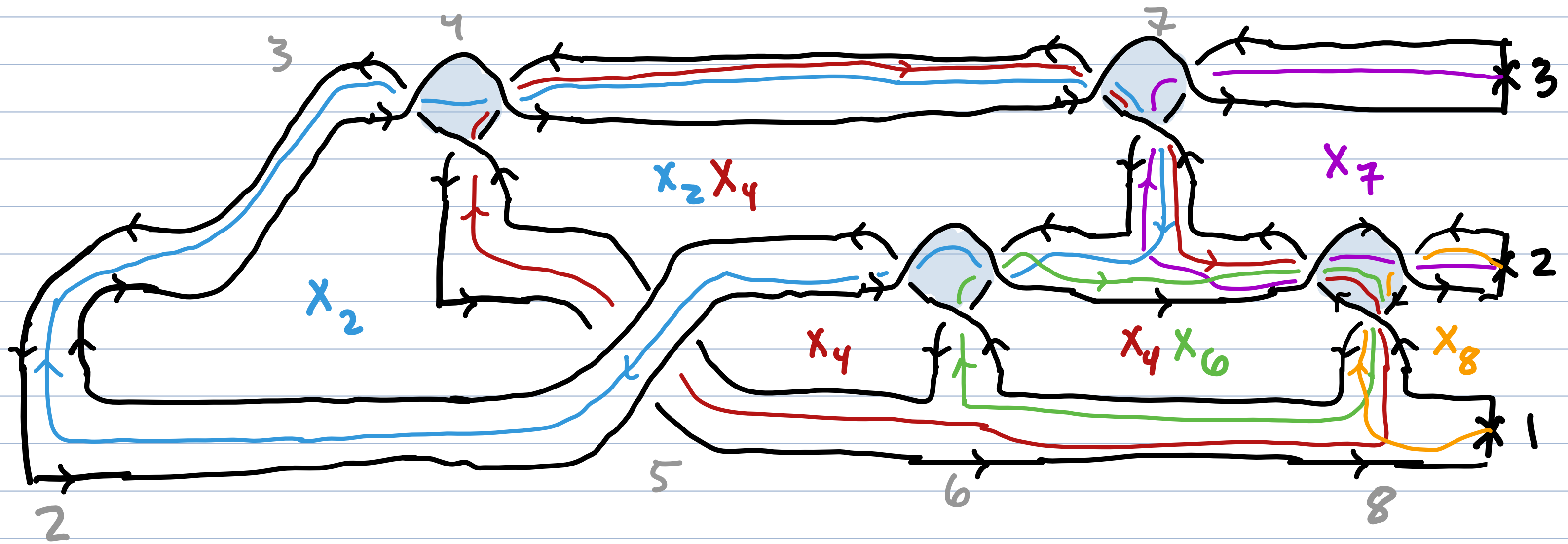
Another way to obtain quiver: from curves on surface

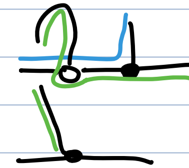
Turn 3D plabic gr. into orientable surface w/ oriented bdry, marked pts via local replacement

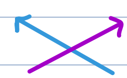




$C_d := \partial S_d$ is (relative) cycle on surface, i.e. it is a cycle or a union of paths ending @ marked pts

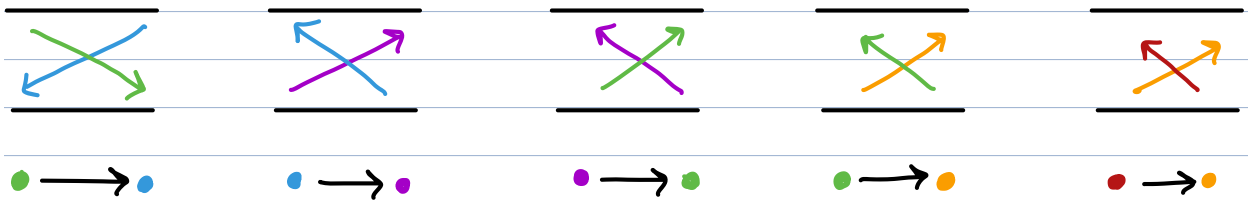


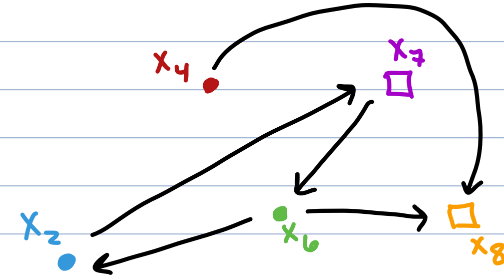


Quiver arrows: intersection on "front" of surface  ⇝ $\bullet \longrightarrow \bullet$ (arrow goes )

In other words, # arrows $x_a \to x_b$ = signed intersection # of C_a & C_b (assuming either a or b mutable)

In ex above:





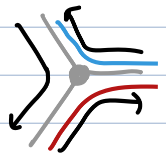
Pf of equivalence of the 2 constructions: Maybe supplemental exercise?

Fix x_a, x_b. Consider C_a, C_b as subsets of G_β. $C_a \cap C_b$ is disj. union of paths.

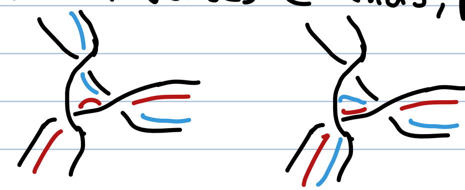
Can draw C_a, C_b so they have @ most 1 intersection pt along ea. path & no other intersec.

Consider a path $(p_0, \dots, p_r)$ which is a connected comp. of $C_a \cap C_b$. How C_a & C_b intersect depends only on: behavior @ ends & color of p_0, p_r

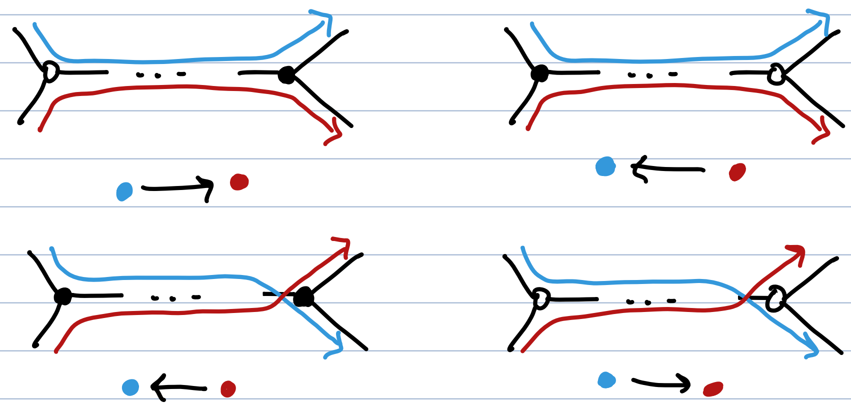
Black vt preserve order of cycles



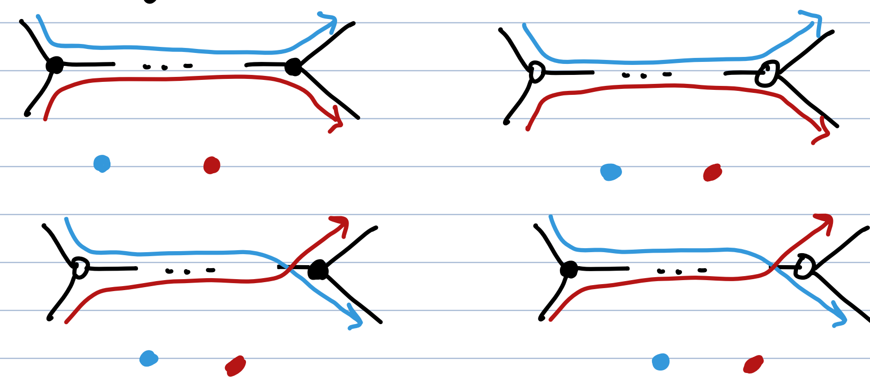
White vt reverses @ ends, preserves in middle

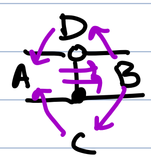


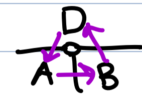
Contributions to Q_β from int rule:

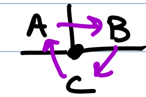


reorienting a cycle reverses arrow



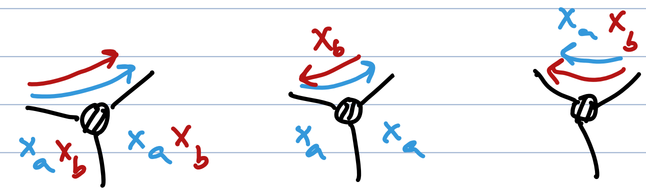
Now for half-arrow construction: 

is the same as 

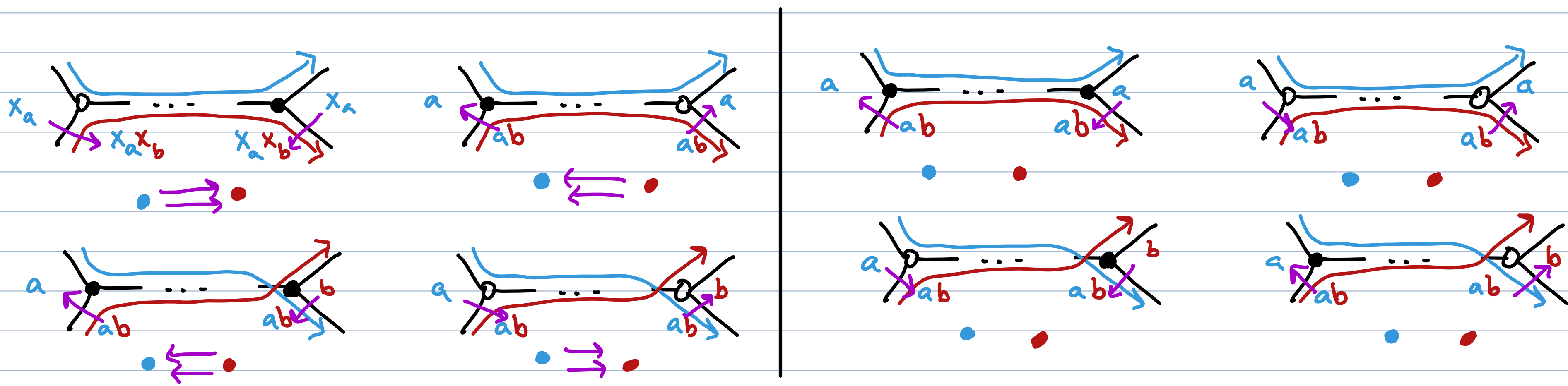
, 

so we compute contribution from ea. trivalent v separately. Easy to check: if x_a appears in all 3 regions, trivalent contributes no arrows involving x_a to $Q_{\underline{\beta}}$. So may assume x_a, x_b appear in some but not all regions $\Rightarrow$ v is along path in $C_a \cap C_b$.

<u>Note:</u> x_a appears to R of C_a.

If v is internal vt of path, contributes no arrows: 

So just need to check contribution from ends:



& check that reorienting a cycle reverses the arrow (easy).

<u>Thm:</u> Let $\underline{\beta}$ be pos. braid word for β. Then

- $Q_{\underline{\beta}}$ sink-recurrent, full-rk
- $\mathbb{C}[X(\underline{\beta})] = \mathcal{U}(\underline{x}_{\underline{\beta}}, Q_{\underline{\beta}}) = \mathcal{A}(\underline{x}_{\underline{\beta}}, Q_{\underline{\beta}})$.

Pf strategy: ⓪ True if β = red. word for w_0. Now assume β not reduced.

① Show validity of thm is unchanged under the following moves:

(C) commutation $\quad \beta = \cdots ij \cdots \longleftrightarrow \beta' = \cdots ji \cdots \quad |i-j|>1$ ← does nothing

(B) braid $\quad \beta = \cdots iji \cdots \longleftrightarrow \beta' = \cdots jij \cdots \quad |i-j|=1$ ← mutation or nothing

(R) rotation $\quad \beta = \underline{\gamma}\, i \longleftrightarrow \beta' = i^* \underline{\gamma} \quad$ where $i^* = n-i$ ← reverse arrows at a frozen, then seq. of mutations

next lec.

Using ⓒ, Ⓑ, can make $\beta = \gamma' i i \underline{\gamma}''$ since β not reduced. Using (R), can make $\beta = \underline{\gamma}\, ii$.

② Induction assuming $\beta = \underline{\gamma}\, ii$.

More on ①:

(C) Iso $\quad X(\beta) \xrightarrow{\sim} X(\beta')$ $\quad$ sends $T_\beta \xrightarrow{\sim} T_{\beta'}$, $V_d' \xrightarrow{\sim} V_d'$.

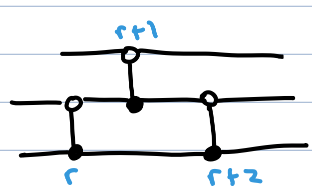
$\cdots F^{r-1} \xrightarrow{i} F^{r+1} \xrightarrow{j} F^r \cdots \mapsto \cdots F^{r-1} \xrightarrow{j} F^{r+1} \xrightarrow{i} F^r \cdots$

This amounts to relabeling cluster var & quiver vt. ✓

(B) • First check quivers: Q_β vs. $Q_{\beta'}$.

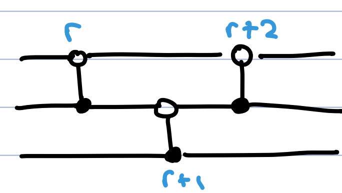
Exercise: If any letters involved in braid move are hollow, quivers differ only by relabelling.

If all letters solid,



G_β

vs.



$G_{\beta'}$

then $Q_{\beta'} = \mu_r(Q_\beta)$ + relabel $r+1 \leftrightarrow r+2$

(Key observation: $S_r =$ 

$= S'_r$)

- Now check variables.

Iso $\cdots$ $F^{r-1} = F_\bullet(A) \xrightarrow{i} F_\bullet(A B_i(z_{r-1})) \xrightarrow{i+1} F_\bullet(A B_i(z_{r-1}) B_{i+1}(z_r)) \xrightarrow{i} F_\bullet(\overbrace{A B_i(z_{r-1}) B_{i+1}(z_r) B_i(z_{r+1})}^{A'})$

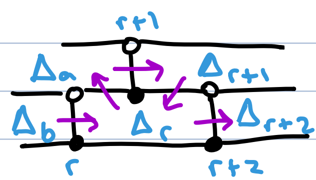
$$\downarrow$$

$F^{r-1} = F_\bullet(A) \xrightarrow{i+1} F_\bullet(A B_{i+1}(z_{r+1})) \xrightarrow{i} F_\bullet(A B_{i+1}(z_{r+1}) B_i(z_{r-1}z_{r-1}-z_r)) \xrightarrow{i+1} F_\bullet(\underbrace{A B_{i+1}(z_{r+1}) B_i(z_{r-1}z_{r+1}-z_r) B_{i+1}(z_{r-1})}_{=A'})$

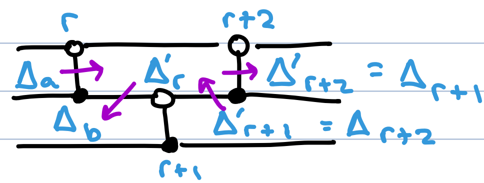
Need to show that x'_r in $(\underline{x}_{\mathfrak{p}'}, Q_{\mathfrak{p}'})$ is mutation of x_r in $(\underline{x}_{\mathfrak{p}}, Q_{\mathfrak{p}})$, $x'_{r+1} = x_{r+2}$, $x'_{r+2} = x_{r+1}$ & all other $x'_c = x_c$.

If $c < r$, then $x'_c = x_c$ since chamber minors & soap films are all the same left of $\underset{r}{\updownarrow}$

<u>Lemma</u>: In this scenario,



$G_{\mathfrak{p}}$



$G_{\mathfrak{p}'}$

1) we have $\Delta_r \Delta'_r = \Delta_a \Delta_{r+2} + \Delta_b \Delta_{r+1}$.

2) for all $c < r$ in $J_{\mathfrak{p}}$, write $\mathrm{ord}_c(\Delta_i) = \mathrm{ord}_{v_c}\Delta_i = \begin{cases} 1 & \text{if } S_c \text{ covers } \Delta_i \\ 0 & \text{else} \end{cases}$

$$\mathrm{ord}_c(\Delta_r) + \mathrm{ord}_c(\Delta'_r) = \min(\mathrm{ord}_c(\Delta_a) + \mathrm{ord}_c(\Delta_{r+2}), \mathrm{ord}_c(\Delta_b) + \mathrm{ord}_c(\Delta_{r+1})) =: q_c$$

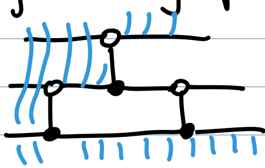
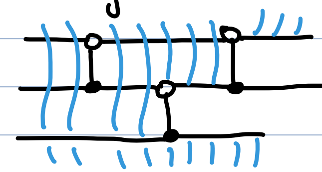
<u>Pf sketch</u>:

1) $\Delta_a = \Delta_{\omega[i+1],[i+1]}(A)$ $\quad\Delta_b = \Delta_{\omega[i],[i]}(A)$ $\quad\Delta_r = \Delta_{\omega[i],[i]}(A B_i(z_{r-1})) = z_{r-1}\Delta_b + \Delta_{\omega[i],s_i[i]}(A)$

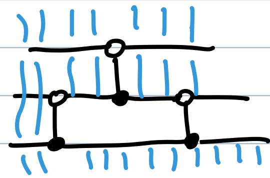
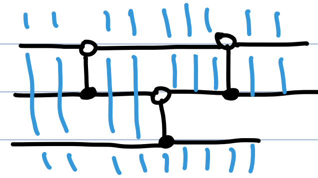
$\Delta_{r+1} = \Delta_{\omega[i+1],[i+1]}(A')$ $\quad\Delta_{r+2} = \Delta_{\omega[i],[i]}(A')$ $\quad\Delta'_r = \Delta_{\omega[i+1],[i+1]}(A B_{i+1}(z_{r+1})) = z_{r+1}\Delta_a + \Delta_{\omega[i+1]s_{i+1}[i+1]}(A)$

Easy to work out effect of mult. by $B_i(z)$ on minors, so can check reln directly by writing all in terms of A, z_{r-1}, z_r, z_{r+1}.

2) Case-by-case analysis using propagation rules. E.g.

 vs. 

$$\operatorname{ord}_c(\Delta_r) + \operatorname{ord}_c(\Delta_r') = \min(\operatorname{ord}_c \Delta_a + \operatorname{ord}_c \Delta_{r+2}, \operatorname{ord}_c \Delta_b + \operatorname{ord}_c \Delta_{r+1})$$
$$0 + 1 = \min(1 + 0, 1 + 0) \quad \checkmark$$

 vs. 

$$\operatorname{ord}_c(\Delta_r) + \operatorname{ord}_c(\Delta_r') = \min(\operatorname{ord}_c \Delta_a + \operatorname{ord}_c \Delta_{r+2}, \operatorname{ord}_c \Delta_b + \operatorname{ord}_c \Delta_{r+1})$$
$$0 + 1 = \min(1 + 0, 1 + 1) \quad \checkmark$$

$\blacksquare$

Lemma $\Rightarrow$ $\Delta_r \Delta_r' = \Delta_a \Delta_{r+2} + \Delta_b \Delta_{r+1}$ can be written as

$N x_r x_r' = N(m + m')$ where $N = \prod_c x_c^{g_c}$, $m = \dfrac{\Delta_a \Delta_{r+2}}{N}$, $m' = \dfrac{\Delta_b \Delta_{r+1}}{N}$.

Note $m = \prod_{\substack{c \leftarrow r \\ \text{in } Q_{\mathfrak{f}}}} x_c$ & $m' = \prod_{\substack{c \to r \\ \text{in } Q_{\mathfrak{f}}}} x_c$, bc. dividing by N exactly captures which arrows cancel in $Q_{\mathfrak{f}}$ (if x_c appears in both $\Delta_a \Delta_{r+2}$ & $\Delta_b \Delta_{r+1}$, then get $x_r \to x_c$ & $x_r \leftarrow x_c$, will cancel).

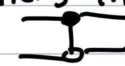
So $\dfrac{\Delta_r \Delta_r'}{N} = \dfrac{\Delta_a \Delta_{r+2} + \Delta_b \Delta_{r+1}}{N}$ is exactly the exchange rel for x_r, & we obtain

$x_r x_r' = m + m'$ $\Rightarrow$ $x_r' =$ mutation of x_r, as desired.

Statement abt remaining cluster var follows from:

- $\Delta'_{r+1} = \Delta_{r+2}$, $\Delta'_{r+2} = \Delta_{r+1}$ & $\Delta'_c = \Delta_c$ for $c > r+2$

- Soap films on G_β, $G_{\beta'}$ agree away from strip around bridges $r, r+1, r+2$.

(R) I won't talk about! Requires a bit more technology – allowing "blue" letters in braid word, expanding defn of Deodhar torus & hypersurface, bridges like  in G_β.

w/ the extra technology, get more moves:

$i\,\underline{\gamma} \longleftrightarrow i^*\underline{\gamma}$ nothing

$\dots i\, j \dots \longleftrightarrow \dots j\, i \dots$ either nothing or mutation

$\underline{\gamma}\, i \longleftrightarrow \underline{\gamma}\, i$ changes final frozen x_r to $1/x_r$ & reverses arrows touching x_r.

(R) is a composition of these moves