Lecture 1: Double Bott-Samelson varieties

Let β be a braid word. So far, we have

- Defined a distinguished torus, the Deodhar torus T_β.

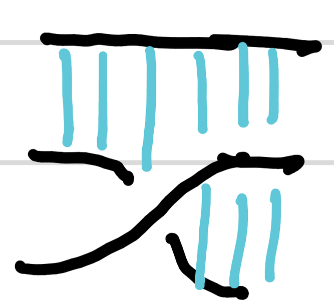
$$T_\beta := \{(\mathcal{F}^0, \dots, \mathcal{F}^\ell) \in X(\beta) \mid \mathcal{F}^0 \xrightarrow{\delta(\beta^j)} \mathcal{F}^j \ \forall j\}$$

- Found a system of coordinates on T_β:

$$\Delta_j := \Delta_{\delta(\beta_j)[i_j],[i_j]}(B_{\beta_j}(z_1, \dots, z_j)), \quad j \in J_\beta := \{i \mid \delta(\beta_{i-1}) = \delta(\beta_i)\}$$

Known as the chamber minors.

- Decomposed the chamber minors into irreducible components x_j, $j \in J_\beta$, using soap films in 3D plabic graphs.

Note that the soap films start getting complicated when they go through a hollow crossing . So today we study the case when this doesn't happen.

For this, we require all hollow crossings to be to the left of all solid crossings. Since $\delta(\beta) = w_0$, this happens if and only if

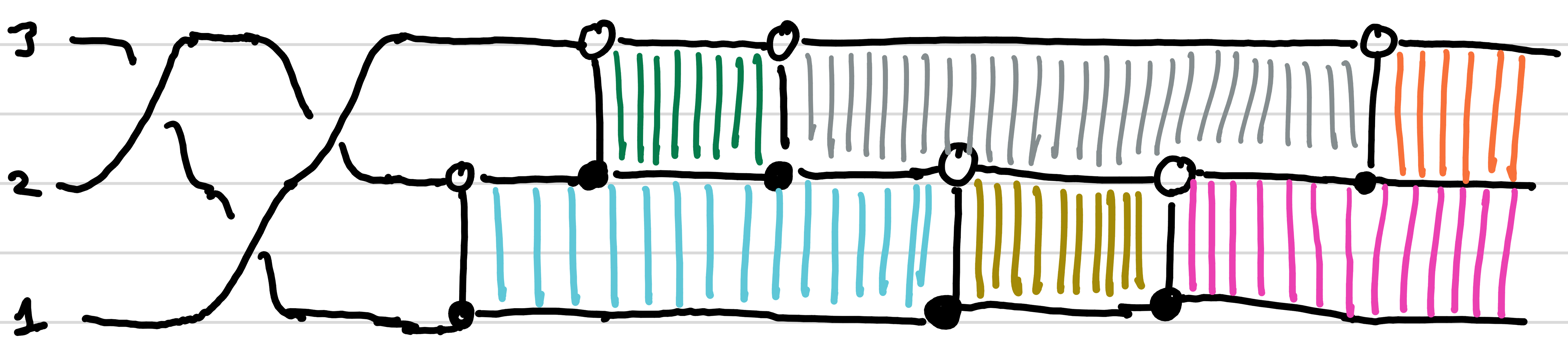
$$\beta = \underline{\Delta} \cdot \gamma$$

where $\underline{\Delta}$ is a minimal braid lift of w_0

(e.g. $\underline{\Delta} = \big((n-1)\cdots 2\,1\big)\big((n-1)\cdots 2\big)\cdots\big((n-1)(n-2)\big)(n-1)\big)$

Note The hollow crossings of $\underline{\Delta}\underline{\gamma}$ are the crossings of $\underline{\Delta}$, and the solid ones are those of $\underline{\gamma}$).

Example $\beta = 212 \cdot 122112$. The 3D plabic graph w/ its soap films is



No region is covered by more than one soap film

$\Rightarrow$ Grid minors are irreducible:

$$\boxed{x_j = \Delta_j \ \forall j \in J_\beta}$$

As it turns out, the varieties $X(\Delta\gamma)$ have already been studied (from a cluster-theoretic point of view) by other authors (Elek-Liu, Goodearl-Yakimov, Shen-Weng)

Def Let $\gamma = \sigma_{i_1} \cdots \sigma_{i_\ell}$. The double Bott-Samelson variety $BS(\gamma)$ is

$$BS(\gamma) := \{ (\mathcal{F}^0 = \mathcal{F}^{std}, \mathcal{F}^1, \dots, \mathcal{F}^\ell) \in \mathcal{F}l(n)^{\ell+1} \mid \mathcal{F}^0 \xrightarrow{s_{i_1}} \mathcal{F}^1 \to \cdots \xrightarrow{s_{i_\ell}} \mathcal{F}^\ell \underbrace{\xrightarrow{w_0} \mathcal{F}^{ant}}_{\substack{\text{Very different} \\ \text{from the braid vty} \\ \text{def-n!}}} \}$$

Thm $BS(\gamma)$ is (isomorphic to) a principal open set in $\mathbb{C}^\ell$.

In particular, it's smooth

Pf Parametrize the flags by $\mathcal{F}(B_{i_1}(z_1))$, $\mathcal{F}(B_{i_1}(z_1)B_{i_2}(z_2))$, ... as we do with braid varieties.

The condition $\mathcal{F}^\ell \xrightarrow{w_0} \mathcal{F}^{ant}$ is equivalent to the nonvanishing of the principal minors of $B_\gamma(z_1, \dots, z_\ell)$.

In fact, we have

Thm $BS(\gamma) \cong X(\Delta\gamma)$

Pf 1 (Using flags) Let $(\mathcal{F}^0, \dots, \mathcal{F}^{l(w_0)}, \mathcal{F}^{l(w_0)+1}, \dots, \mathcal{F}^{l(w_0)+l}) \in X(\Delta\gamma)$. Note that

$\mathcal{F}^{std} \xrightarrow{w_0} \mathcal{F}^{l(w_0)}$, so $\mathcal{F}^{l(w_0)} = \mathcal{F}(M)$ with $M \in B w_0 B$, ie $M = U_1 w_0 U_2$, and U_1 is unique provided it has 1's on the diagonal. Now $(w_0 U_1^{-1} \mathcal{F}^{l(w_0)}, w_0 U_1^{-1} \mathcal{F}^{l(w_0)+1}, \dots, w_0 U_1^{-1} \mathcal{F}^{l(w_0)+l}) \in BS(\gamma)$.

Conversely, if $(\mathcal{F}^0, \dots, \mathcal{F}^l) \in BS(\gamma)$ then $\mathcal{F}^l = \mathcal{F}(N)$ with $N \in (w_0 B w_0) B$, ie
$N = (w_0 V_1 w_0) V_2$, and V_1 is unique provided it has 1's on the diagonal. Translate by $V_1^{-1} w_0$, and complete the chain in a unique way to an element of $X(\Delta\gamma)$.

Pf 2 (Using matrices) Observe that

$$B_\Delta(z_1, \dots, z_{\binom{n}{2}}) = \begin{pmatrix} z_{n-1} & -z_{2n-3} & & (-1)^{k-2} z_{\binom{n}{2}} & (-1)^{k-1} \\ \vdots & \vdots & & (-1)^{k-2} & 0 \\ z_2 & -z_n & \iddots & & \\ z_1 & -1 & & 0 & \vdots \\ 1 & 0 & & 0 & 0 \end{pmatrix} = w_0 L \text{ for a lower } \Delta' r \text{ matrix } L.$$

Use this L to translate as in the first proof above

Using this, we have a nice expression for the candidate cluster variables

We have the iso $\varphi: BS(\gamma) \to X(\Delta\gamma)$, $\varphi(z_1,\dots,z_\ell) = (p_1,\dots,p_{\binom{n}{2}}, z_1,\dots,z_\ell)$ for uniquely def-d functions $p_1,\dots,p_\ell$. For $j=1,\dots,\ell$ we have the variable

$$x_j = \Delta_{w_0[i_j],[i_j]} B_{\Delta\sigma_{i_1}\dots\sigma_{i_j}}(p_1,\dots,p_{\binom{n}{2}}, z_1,\dots,z_j) =$$

$$\Delta_{w_0[i_j],[i_j]} B_\Delta(p_1,\dots,p_{\binom{n}{2}}) B_{\gamma_j}(z_1,\dots,z_j) =$$

$$\Delta_{w_0[i_j],[i_j]} w_0 L(\varphi) B_{\gamma_j}(z_1,\dots,z_j) = \Delta_{[i_j],[i_j]} L(\varphi) B_{\gamma_j}(z_1,\dots,z_j) =$$

$$\Delta_{[i_j],[i_j]} B_{\gamma_j}(z_1,\dots,z_j)$$

Thus, the "cluster var-s" of $BS(\gamma)$ are the principal minors of braid matrices associated to partial products $\gamma_1,\dots,\gamma_\ell=\gamma$.

Example $\gamma = 11212$, $BS(\gamma) \cong X(212(11212))$

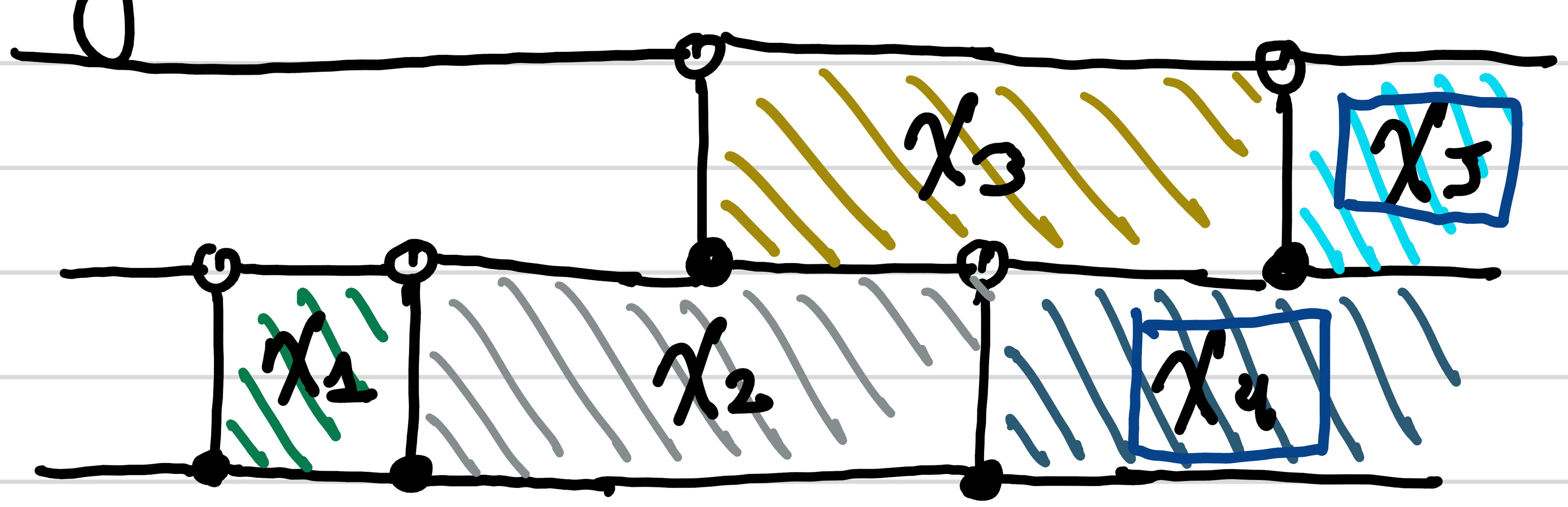
$$\chi_1 = \Delta_{1,1}\begin{pmatrix} z_1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = z_1, \qquad \chi_2 = \Delta_{1,1}\begin{pmatrix} z_1z_2-1 & -z_1 & 0 \\ z_2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = z_1z_2-1$$

$$\chi_3 = \Delta_{\{1,2\},\{1,2\}}\begin{pmatrix} z_1z_2-1 & -z_1z_3 & z_1 \\ z_2 & -z_3 & 1 \\ 0 & 1 & 0 \end{pmatrix} = z_3 \qquad \chi_4 = \Delta_1\begin{pmatrix} z_1z_2z_4 - z_1z_3 - z_4 & -z_1z_2+1 & z_1 \\ z_2z_4 - z_3 & -z_2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= z_1z_2z_4 - z_1z_3 - z_4$$

$$\chi_5 = \Delta_{\{1,2\},\{1,2\}}\begin{pmatrix} z_1z_2z_4 - z_1z_3 - z_4 & -z_1z_2z_5 + z_1 + z_5 & z_1z_2-1 \\ z_2z_4 - z_3 & -z_2z_5+1 & z_2 \\ 1 & 0 & 0 \end{pmatrix} = z_3z_5 - z_4$$

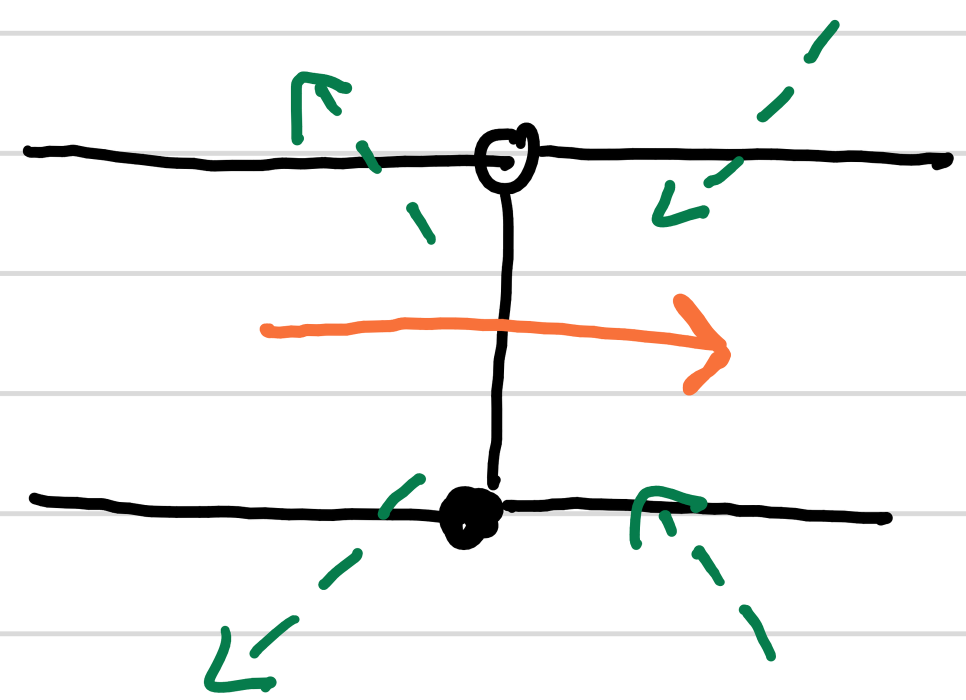
We'll also forget the hollow X-ings in the 3D plabic graphs of $\Delta\gamma$, since they don't give any extra info



Frozen variables: Regions unbounded on the right.

Note These are the principal minors of Δ_γ

Quiver Around each bridge, have the following configuration



$\longrightarrow$ = full arrow

$\dashrightarrow$ = half arrow

add # of half arrows: will get an integer if one of the regions is mutable.

Two half arrows in opposite directions cancel.

Thm (Shen-Weng) This defines a cluster structure on $\mathbb{C}[BS(\gamma)]$

The proof follows the same basic steps as the proof we sketched on Monday

- $\mathbb{C}[BS(\gamma)]$ is a normal noetherian domain because it's the localization of a polynomial algebra: $\mathbb{C}[BS(\gamma)] = \mathbb{C}[z_1, \dots, z_\ell][\Delta_{[i],[i]}(B_\gamma(z_1, \dots, z_\ell))^{-1}, i=1, \dots, n-1]$
- We've seen that the candidate cluster variables are irreducible

Now we want to verify that, upon mutating, we get a regular function on $BS(\beta)$. We'll use the following result

Lemma ⓐ For any matrix A, $i = 1, \dots, k-1$ and $z, w \in \mathbb{C}$

$$\Delta_{[i],[i]}(AB_i(z))\Delta_{[i],[i]}(B_i(w)A) - \Delta_{[i],[i]}(A)\Delta_{[i],[i]}(B_i(w)AB_i(z)) = \Delta_{[i-1],[i-1]}(A)\Delta_{[i+1],[i+1]}(A)$$

This can be proven using the following

Lemma (5) $\Delta_{[i],[i]}(AB_i(z)) = z\Delta_{[i],[i]}(A) - \Delta_{[i],S_i[i]}(A)$

$$\Delta_{[i],[i]}(B_i(w)A) = w\Delta_{[i],[i]}(A) + \Delta_{S_i[i],[i]}(A)$$

Lemma (5) is an easy consequence of Cauchy-Binet. For Lemma (4), thanks to Lemma (5) the lhs is

$$(z\Delta_{[i],[i]}(A) - \Delta_{[i],S_i[i]}(A))(w\Delta_{[i],[i]}(A) + \Delta_{S_i[i],[i]}(A))$$

$$- \Delta_{[i],[i]}(A)(w\Delta_{[i],[i]}(AB_i(z)) + \Delta_{S_i[i],[i]}(AB_i(z))) =$$

$$\cancel{zw\Delta_{[i],[i]}(A)^2} + \cancel{z\Delta_{[i],[i]}(A)\Delta_{S_i[i],[i]}(A)} - \cancel{w\Delta_{[i],[i]}(A)\Delta_{[i],S_i[i]}(A)} - \Delta_{[i],S_i[i]}(A)\Delta_{S_i[i],[i]}(A)$$

$$-\Delta_{[i],[i]}(A)(\cancel{zw\Delta_{[i],[i]}(A)} - \cancel{w\Delta_{[i],S_i[i]}(A)} + \cancel{z\Delta_{S_i[i],[i]}(A)} - \Delta_{S_i[i],S_i[i]}(A))$$

$$= \Delta_{[i],[i]}(A)\Delta_{S_i[i],S_i[i]}(A) - \Delta_{[i],S_i[i]}(A)\Delta_{S_i[i],[i]}(A)$$

$$= \det\begin{pmatrix}\Delta_{[i],[i]}(A) & \Delta_{S_i[i],[i]}(A)\\ \Delta_{[i],S_i[i]}(A) & \Delta_{S_i[i],S_i[i]}(A)\end{pmatrix} \quad (*)$$

Assume $\Delta_{[i-1],[i-1]}(A) = 0$, but $\Delta_{[j],[j]}(A) \neq 0$ $\forall j < i-1$. We may do elementary column operations so that A has the form

$$\begin{pmatrix} a_{11} & \cdots & a_{1,i-2} & 0 & a_{1,i} & a_{1,i+1} & \\ a_{21} & \cdots & a_{2,i-2} & 0 & a_{2,i} & a_{2,i+1} & \\ \vdots & & & \vdots & \vdots & \vdots & \\ a_{i-1,1} & \cdots & a_{i-1,i-2} & 0 & a_{i-1,i} & a_{i-1,i+1} & \\ a_{i,1} & \cdots & a_{i,i-2} & a_{i,i-1} & a_{i,i} & a_{i,i+1} & \\ a_{i+1,1} & \cdots & a_{i+1,i-2} & a_{i+1,i-1} & a_{i+1,i} & a_{i+1,i+1} & \cdots \end{pmatrix}$$

$B_i(w)$

and we see that
$$\Delta_{[i],[i]}(A) = (-1)^{2i-1} a_{i,i-1} \Delta_{[i-1],s_{i-1}[i-1]}(A)$$
$$\Delta_{[i],s_i[i]}(A) = (-1)^{2i-1} a_{i,i-1} \Delta_{[i-1],\{1,\dots,i-2,i+1\}}(A)$$
$$\Delta_{s_i[i],[i]}(A) = (-1)^{2i} a_{i+1,i-1} \Delta_{[i-1],s_i[i-1]}(A)$$
$$\Delta_{s_i[i],s_i[i]}(A) = (-1)^{2i} a_{i+1,i-1} \Delta_{[i-1],\{1,\dots,i-2,i+1\}}(A)$$

So $(*)=0$. Similarly, if $\Delta_{[i+1],[i+1]}(A) = 0$ then in an open set $(*)=0$. So $\Delta_{[i-1],[i-1]}\Delta_{[i+1],[i+1]}$ divides $(*)$, and the result follows by looking at the coeff. of a single monomial

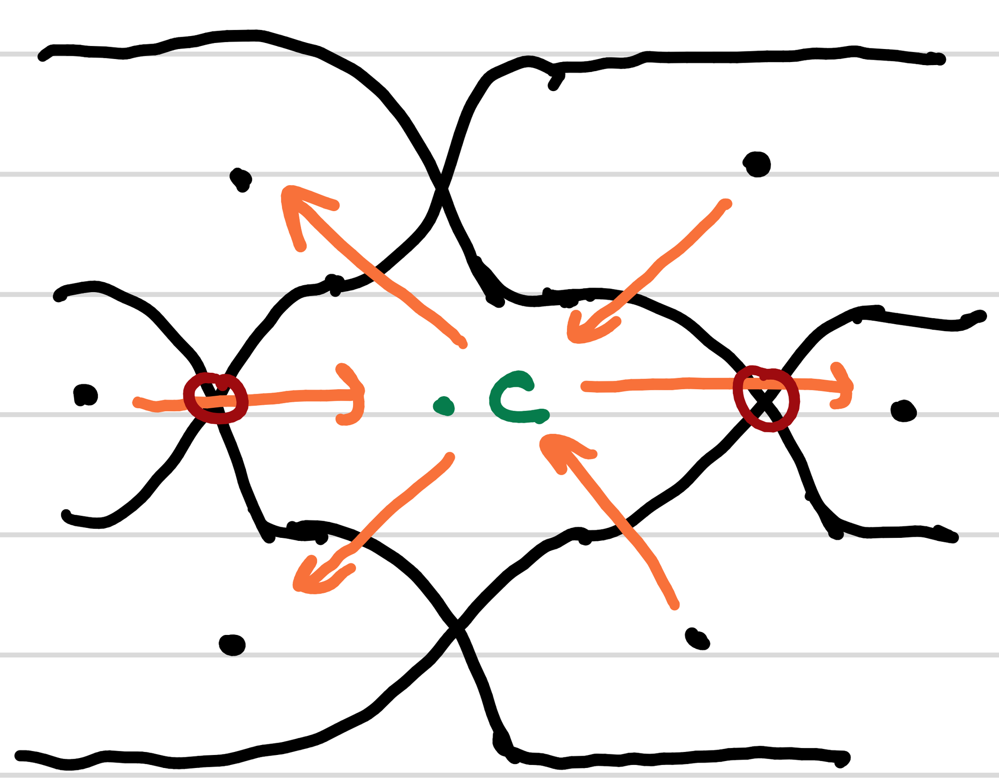
Armed with this, we show:

Lemma (Half of Starfish lemma). Let c be a mutable vertex. Then

$$x_c' := \frac{\prod_{e \to c} x_e + \prod_{c \to d} x_d}{x_c}$$

is a regular function on $BS(\gamma)$.

Pf Clearly x_c' is regular outside of the vanishing locus of x_c. Let us now look at the quiver in more detail. The more complicated case is when we have c of degree 6.



If $x_c = \Delta_{[i],[i]}(A)$ then the monomial $\prod_{c \to d} x_d$ has the form

$$\Delta_{[i+1],[i+1]}(A)\,\Delta_{[i-1],[i-1]}(A) \cdot \Delta_{[i],[i]}(AX) \quad (\heartsuit)$$

and the monomial $\prod_{e \to c} x_e$ has the form

$$\Delta_{[i+1],[i+1]}(AY)\,\Delta_{[i-1],[i-1]}(AY) \cdot \Delta_{[i],[i]}(A B_i^{-1}(z_c)) \quad (\diamondsuit)$$

By Lemma Ⓐ we may write $(\heartsuit)$ as

$$\left(\Delta_{[i],[i]}(AB_i(\mathfrak{z}))\cdot\Delta_{[i],[i]}(B_i(\varpi)A) - \Delta_{[i],[i]}(A)\Delta_{[i],[i]}(B_i(\varpi)A\,B_i(\mathfrak{z}))\right)\Delta_{[i],[i]}(AX)$$

and we may write $(\diamondsuit)$ as

$$\left(\Delta_{[i],[i]}(AYB_i(z))\cdot\Delta_{[i],[i]}(B_i(w)AY) - \Delta_{[i],[i]}(AY)\Delta_{[i],[i]}(B_i(w)AYB_i(z))\right)\cdot\Delta_{[i],[i]}(AB_i^{-1}(z_c))$$

Now, $\Delta_{[i],[i]}(AY)=\Delta_{[i],[i]}(A)$ because there is no s_i between the two marked crossings. Perhaps moving to an open subset of the vanishing locus of x_c, we may find $\mathfrak{z},\varpi,z,w$ so that

$$\Delta_{[i],[i]}(AB_i(\mathfrak{z}))\Delta_{[i],[i]}(B_i(\varpi)A)\Delta_{[i],[i]}(AX) + \Delta_{[i],[i]}(AYB_i(z))\Delta_{[i],[i]}(B_i(w)AY)\Delta_{[i],[i]}(AB_i^{-1}(z_c)) = 0$$

And in this locus

$$x'_c = -\Delta_{[i],[i]}(B_i(\varpi)AB_i(\mathfrak{z})) - \Delta_{[i],[i]}(B_i(w)AYB_i(z))$$

is regular. So χ'_c is regular outside of codim 2 and thus it is regular on $BS(\gamma)$.