WARTHOG. Monday, Lecture I.

Flags and relative positions

* Notations and conventions

- We'll always work / $\mathbb{C}$.
- Fix $n>0$. We denote:
  - $G = GL_n$
  - $B \subseteq G$, subgroup of invertible upper triangular matrices
  - S_n, the symmetric group on n elements.

$$S_n = \langle s_1, \dots, s_{n-1} \mid s_i^2 = 1,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ s_i s_j = s_j s_i \text{ if } |i-j|>1 \rangle$$

s_i is the transposition $s_i = (i\ \ i+1)$

it has a length function $\ell(w) =$ minimal possible length of a word in generators $s_1, \dots, s_{n-1}$ yielding w

$= \#\{(i,j) \mid i<j \text{ but } w(i) > w(j)\}$

Longest element: $w_0 = [n, n-1, \dots, 2, 1] = (s_1 \dots s_{n-1})(s_1 \dots s_{n-2}) \dots (s_1 s_2) s_1$

$$\ell(w_0) = \binom{n}{2}$$

* Flags

Def A (complete) flag in $\mathbb{C}^n$ is a chain of subspaces

$$\mathcal{F}_\bullet = (\{0\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_n = \mathbb{C}^n)$$

st $\dim \mathcal{F}_i = i \;\forall\, i = 0, \dots, n$.

The set of all complete flags is a proj-ve. alg. vty. called the flag variety $\mathcal{F}\ell(n)$.

Any matrix $M = (m_1 | \dots | m_n) \in G$ defines a flag $\mathcal{F}_\bullet(M)$

$$\mathcal{F}_i(M) = \operatorname{span}\{m_1, \dots, m_i\}$$

Note that any flag is of the form $\mathcal{F}_\bullet(M)$ for some $M \in G$, and

$$\mathcal{F}_\bullet(M) = \mathcal{F}_\bullet(N) \iff \exists\; U \in B \text{ s.t. } M = NU.$$

so we get an identification $\mathcal{F}\ell(n) \cong G/B$.

Note that $G \curvearrowright \mathcal{F}\ell(n)$, via $(g\mathcal{F})_i = g(\mathcal{F}_i)$. This action is transitive and B is the stabilizer of the standard flag $\mathcal{F}^{std} = \mathcal{F}_\bullet(I_n)$.

* Special flags

For $w \in S_n$, we consider its permutation matrix

$$w_{ij} = \begin{cases} 1, & \text{if } i = w(j) \\ 0, & \text{else} \end{cases}, \quad \text{i.e.} \quad w = \left(e_{w(1)} \middle| \cdots \middle| e_{w(n)} \right)$$

and the corresponding flag $\mathcal{F}_\bullet(w)$.

In particular, we have the aforementioned standard flag

$$\mathcal{F}_\bullet^{std} = \mathcal{F}_\bullet(e) = \left(0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \cdots \subseteq \langle e_1, \ldots, e_n \rangle = \mathbb{C}^n \right)$$

and the antistandard flag

$$\mathcal{F}_\bullet^{ant} = \mathcal{F}_\bullet(w_0) = \left(0 \subseteq \langle e_n \rangle \subseteq \langle e_n, e_{n-1} \rangle \subseteq \cdots \subseteq \langle e_n, e_{n-1}, \ldots, e_1 \rangle = \mathbb{C}^n \right)$$

Note that

$$\dim\left(\mathcal{F}_i^{std} \cap \mathcal{F}_j(w)\right) = \# \{1, \ldots, i\} \cap \{w(1), \ldots, w(j)\}$$

* Relative position

Def We say that $\mathcal{F}_\bullet, \mathcal{G}_\bullet \in \mathcal{F}\ell(n)$ are in relative position $w \in S_n$ if $\exists\, g \in G$ st
$$g\mathcal{F}_\bullet = \mathcal{F}_\bullet^{std}, \qquad g\mathcal{G}_\bullet = \mathcal{F}_\bullet(w).$$
We'll write $\mathcal{F}_\bullet \xrightarrow{w} \mathcal{G}_\bullet$

Lemma $\mathcal{F}_\bullet \xrightarrow{w} \mathcal{G}_\bullet$ iff $\forall\, i,j$, $\dim(\mathcal{F}_i \cap \mathcal{G}_j) = \#\{1,\dots,i\} \cap \{w(1),\dots,w(j)\}$

So, for any two flags $\mathcal{F}_\bullet, \mathcal{G}_\bullet$ $\exists!\, w \in S_n$ st. $\mathcal{F}_\bullet \xrightarrow{w} \mathcal{G}_\bullet$. $-$

Corollary For $i = 1, \dots, n-1$,
$$\mathcal{F}_\bullet \xrightarrow{s_i} \mathcal{G}_\bullet \text{ if and only if } \mathcal{F}_i \neq \mathcal{G}_i \text{ but } \mathcal{F}_j = \mathcal{G}_j \ \forall\, j \neq i.$$

In terms of matrices:

$\mathcal{F}_\bullet(M) \xrightarrow{w} \mathcal{F}_\bullet(N)$ iff $\exists\, g \in GL(n)$ st. $gM \in B$, $gN \in wB$

iff $\exists\, g \in GL(n)$, $U_1, U_2 \in B$ st $gM = U_1$, $gN = wU_2$

iff $M^{-1}N \in BwB$.

* Schubert cells

Def For $w \in S_n$, the Schubert cell $C_w \subseteq Fl(n)$ is
$$C_w := \{ F_\bullet \in Fl(n) \mid F_\bullet^{std} \xrightarrow{w} F_\bullet \}$$

By the previous discussion, $F_\bullet(M) \in C_w \iff M \in BwB$, that is
$$C_w = BwB/B.$$

So, to get a matrix representing an element of C_w, start with w and multiply on the left by an upper triangular matrix ie, do elementary row ops where you're only allowed to add a multiple of a row to a row above it. We can also do upper triangular column operations, that do not change the flag. It follows that any flag $F_\bullet \in C_w$ can be written as $F_\bullet(M)$ where

$$M_{ij} = \begin{cases} 1, & \text{if } i = w(j) \quad \leftarrow \text{pivots} \\ 0, & \text{if } i > w(j) \quad \leftarrow 0 \text{ to the right of a pivot} \\ 0, & \text{if } j > w^{-1}(i) \quad \leftarrow 0 \text{ below a pivot} \end{cases}$$

. Moreover, M is unique

e.g if $n=5$, $W=[4,2,1,5,3]$

$$\begin{pmatrix} * & * & 1 & 0 & 0 \\ * & 1 & 0 & 0 & 0 \\ * & 0 & 0 & * & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow C_w \cong \mathbb{C}^{l(w)}$$

* Opposite Schubert cells

Def For $w \in S_n$, the *opposite Schubert cell* is

$$C^w := \{ \mathcal{F}_\bullet \in \mathcal{F}\ell(n) \mid \mathcal{F}_\bullet \xrightarrow{w^{-1}w_0} F^{ant} \}$$

Similarly to what's above, $\mathcal{F}_\bullet(M) \in C^w$ iff $M^{-1}w_0 \in B w^{-1} w_0 B$

iff $w_0 M \in B w_0 w B$

iff $M \in (w_0 B w_0) w B$.

So $C^w = w_0 C_{w_0 w}$, and $C^w \cong \mathbb{C}^{\ell(w_0 w)} = \mathbb{C}^{\binom{n}{2} - \ell(w)}$.

In particular,

$\mathcal{F}_\bullet(M) \in C^e \iff \mathcal{F}_\bullet(M) \xrightarrow{w_0} F^{ant} \iff \forall\, i = 1, \dots, n:$

$$\operatorname{span}\{m_1, \dots, m_i\} \cap \operatorname{span}\{e_{i+1}, \dots, e_n\} = \{0\}$$

$\iff \Delta_{i,i}(M) \neq 0 \;\; \forall\, i = 1, \dots, n.$

* Braid varieties

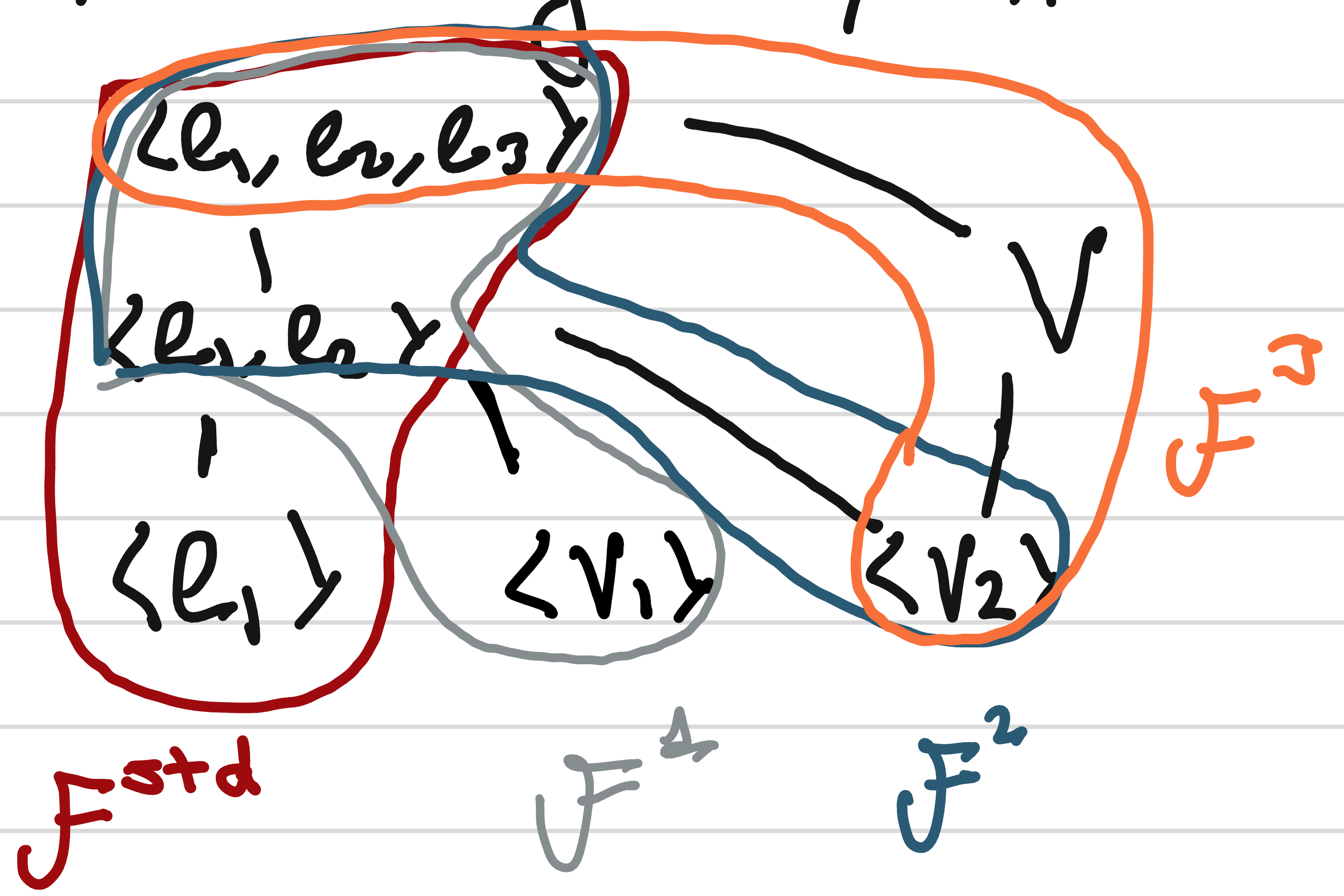
Def Let $\mathbf{i} = (i_1, \dots, i_r)$ be a word in the alphabet $\{1, \dots, n-1\}$. The **braid variety** is

$$X(\mathbf{i}) = \{ \mathcal{F}_\bullet^{std} \xrightarrow{s_{i_1}} \mathcal{F}_\bullet^1 \xrightarrow{s_{i_2}} \mathcal{F}_\bullet^2 \to \dots \xrightarrow{s_{i_r}} \mathcal{F}_\bullet^r = \mathcal{F}_\bullet^{ant} \} \subseteq \mathcal{F}\ell(n)^{r+1}$$

Remarks

1) Why "braid" varieties? See the exercises!

2) $X(\mathbf{i})$ may be $\emptyset$ if $\mathbf{i}$ is not "complicated enough". E.g. $n=3$, $\mathbf{i} = (1,1,2)$



Since $\langle v_2 \rangle \subseteq \langle e_1, e_2 \rangle$, we can't have $\langle v_2 \rangle = \langle e_3 \rangle$, so $\mathcal{F}_\bullet^3 \neq \mathcal{F}_\bullet^{ant}$!

Examples

1) $n=2$; $\mathbf{i}=(1,1)$, $X(\mathbf{i})\cong\mathbb{C}^\times$

2) $n=3$, $\mathbf{i}=(1,2,1)$, $X(\mathbf{i})=\{pt\}$