Lecture 7: main exercises

Exercise 7.1. For each of the following braids:
(a) Draw its inductive weave.
(b) Find its solid crossings.
(c) Determine which Deodhar hypersurfaces are frozen, and which are mutable.

(i) $\beta = \sigma_1^n \in \mathrm{Br}_2^+$. For this braid, also find a recursive formula for the chamber minors.
(ii) $\beta = \sigma_1^2\sigma_2^2\sigma_1^2\sigma_2 \in \mathrm{Br}_3^+$.
(iii) $\beta = \sigma_1\sigma_2\sigma_1\sigma_1^2\sigma_2\sigma_1\sigma_2 \in \mathrm{Br}_3^+$.

Exercise 7.2. In this exercise we will use the Cauchy-Binet formula, that says that if A, B are $n \times n$-matrices and $I, J \subseteq [n]$ are sets of the same size then

$$\Delta_{I,J}(AB) = \sum_{\substack{K \subseteq [n] \\ |K|=|I|=|J|}} \Delta_{I,K}(A)\Delta_{K,J}(B).$$

(a) As a warm-up: what does the Cauchy-Binet formula say when $|I| = |J| = 1$? What does it say when $|I| = |J| = n$?

(b) Let M be any $n \times n$ matrix . Let $j, i = 1, \ldots, n-1$ and let $I \subseteq [n]$ be a subset of size $|I| = i$. Show that:

$$\Delta_{I,[i]}(MB_j(z)) = \begin{cases} \Delta_{I,[i]}(M) & \text{if } j \neq i, \\ z\Delta_{I,[i]}(M) + \Delta_{I,s_i[i]}(M) & \text{if } j = i. \end{cases}$$

and

$$\Delta_{[i],I}(B_j(z)M) = \begin{cases} \Delta_{[i],I}(M) & \text{if } j \neq i, \\ z\Delta_{[i],I}(M) - \Delta_{s_i[i],I}(M) & \text{if } j = i, \end{cases}$$

where $s_i[i] = \{s_i(1), \ldots, s_i(i)\} = \{1, \ldots, i-1, i+1\}$.

Note: To see the general pattern, it may suffice to assume $i = 2$, $n = 4$.

(c) Use part (a) to show that, if $z, w \in \mathbb{C}$ then

$$\Delta_{[i],[i]}(MB_i(z))\Delta_{[i],[i]}(B_i(w)M) - \Delta_{[i],[i]}(M)\Delta_{[i],[i]}(B_i(w)MB_i(z)) = \det\begin{pmatrix} \Delta_{[i],[i]}(M) & \Delta_{s_i[i],[i]}(M) \\ \Delta_{[i],s_i[i]}(M) & \Delta_{s_i[i],s_i[i]}(M) \end{pmatrix}.$$

Note that the right-hand side is independent of z, w! The *Desnanot-Jacobi* identity asserts that

$$\det\begin{pmatrix} \Delta_{[i],[i]}(M) & \Delta_{s_i[i],[i]}(M) \\ \Delta_{[i],s_i[i]}(M) & \Delta_{s_i[i],s_i[i]}(M) \end{pmatrix} = \Delta_{[i-1],[i-1]}(M)\Delta_{[i+1],[i+1]}(M).$$

This identity will prove useful tomorrow.