Lecture 16: main exercises

Exercise 16.1. Consider the quiver $\boxed{a} \to ① \to \boxed{b}$.

(a) Show that its cluster algebra A is isomorphic to $\mathbb{C}[x_1, x_1', y_a^{\pm 1}, y_b^{\pm 1}]/(x_1x_1' - y_a - y_b)$. (cf. Exercise 5.1).

(b) Let $z := y_1^{-1}x_1 \in A$. Show that the following seed defines a cluster structure on A:

$$\boxed{y_a} \qquad ⓩ \to \boxed{y_a^{-1}y_b}$$

Thus, the same algebra A can have many different cluster structures.

Exercise 16.2. Recall that we set

$$x_j(z) = \begin{pmatrix} 1 & \cdots & & & \cdots & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ 0 & \cdots & 1 & z & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 1 \end{pmatrix}, \qquad \dot{s}_j = \begin{pmatrix} 1 & \cdots & & & \cdots & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & \cdots & 0 \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 1 \end{pmatrix}$$

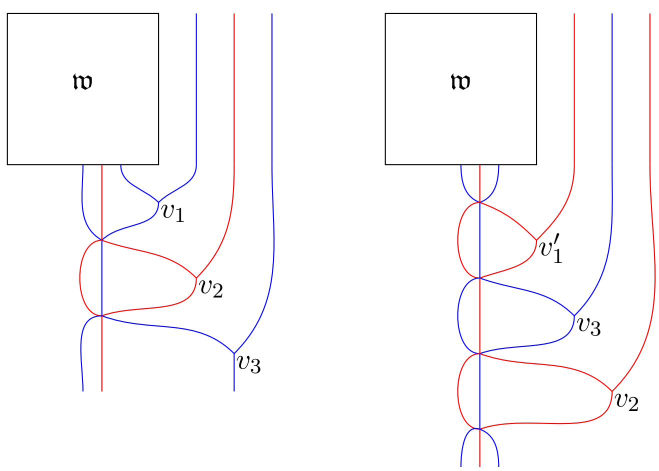
so that $B_j(z) = x_j(z)\dot{s}_j$. Let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced word. Show that we can write

$$B_{i_1}(z_1)\chi_{i_1}(u_1) \cdots B_{i_\ell}(z_\ell)\chi_{i_\ell}(u_\ell) = x_{i_1}(z_1)\dot{s}_{i_1}\chi_{i_1}(u_1) \cdots x_{i_\ell}(z_\ell)\dot{s}_{i_\ell}\chi_{i_\ell}(u_\ell) = g\left(\dot{s}_{i_1}\chi_{i_1}(u_1) \cdots \dot{s}_{i_\ell}\chi_{i_\ell}(u_\ell)\right)$$

where g is an upper uni-triangular matrix.

Hint: Show that if $s_{i_1} \cdots s_{i_\ell}$ is reduced, then $\left(\dot{s}_{i_1} \cdots \dot{s}_{i_{\ell-1}}\right) x_{i_\ell}(z) \left(\dot{s}_{i_1} \cdots \dot{s}_{i_{\ell-1}}\right)^{-1}$ is upper uni-triangular.

Exercise 16.3. Consider the following weaves



Show that the corresponding quivers are mutation equivalent, with mutation at the vertex v_1. Compare to Exercise 11.2 from Thursday's first lecture. Note that this is also similar to Exercise 9.2 in Wednesday's first lecture, but here we are not assuming we are in the double Bott-Samelson case.