Lecture 15: main exercises

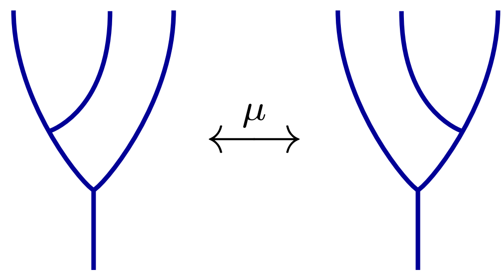
Exercise 15.1. For an integer x denote $[x]_+ = \max(x,0)$ and $[x]_- = \min(x,0)$. Prove that

$$[b-a-c+\min(a,b,c)]_- = -[a+c-b-\min(a,b,c)]_+ = \min(a,b)-c-a+\min(b,c),$$

$$[b-a-c+\min(a,b,c)]_+ = -[a+c-b-\min(a,b,c)]_- = b+\min(a,b,c)-\min(a,b)-\min(b,c)$$

Hint: use tropicalization!

Exercise 15.2. Consider the following figure as part of the bigger weave. Suppose that the Lusztig cycle C_i (not corresponding to trivalent vertices in the picture) has weights a_i, b_i, c_i at the top, and A_i the corresponding cluster variable. Let v_1 (resp. v_2) denote the top (resp. bottom trivalent vertex), A_{v_1} (resp. A_{v_2}) the corresponding cluster variables in the left weave, and $\overline{A_{v_1}}$ (resp. $\overline{A_{v_2}}$) the corresponding cluster variables in the right weave.



(a) In the left weave, prove that

$$A_{v_1} = z_2 \prod A_i^{a_i+b_i-\min(a_i,b_i)}$$

and

$$A_{v_2} = \left(z_2z_3 - \prod A_i^{-2b_i}\right) \prod A_i^{a_i+b_i+c_i-\min(a_i,b_i,c_i)}.$$

(b) In the right weave, prove that:

$$\overline{A_{v_1}} = z_3 \prod A_i^{b_i+c_i-\min(b_i,c_i)}, \quad \text{and} \quad \overline{A_{v_2}} = A_{v_2}.$$

(c) Prove that

$$A_{v_1}\overline{A_{v_1}} = A_{v_2} \prod A_i^{b_i+\min(a_i,b_i,c_i)-\min(a_i,b_i)-\min(b_i,c_i)} + \prod A_i^{a_i+c_i-\min(a_i,b_i)-\min(b_i,c_i)}$$

Use Exercise 15.1 to conclude that A_{v_1} and $\overline{A_{v_1}}$ are related by mutation.