Lecture 10: main exercises

For a braid word $\mathbf{i}$, we consider the double Bott-Samelson variety $\mathrm{BS}(\mathbf{i})$ with the initial seed $(x_{\mathbf{i}}, Q_{\mathbf{i}})$ described in lecture.

Exercise 10.1. Take the braid word on six strands 142145. Note that the letter 3 is missing from this braid word. Show that

$$\mathrm{BS}(142145) \cong \mathrm{BS}(121) \times \mathrm{BS}(445).$$

You could easily do this using cluster algebras, but try to do it without. The upshot is that in many arguments we can assume that every letter in $\{1, 2, \ldots, n-1\}$ actually appears at least once in a braid word.

Exercise 10.2. Assume $\mathbf{i} = \cdots i\underline{i}i \cdots$. Show explicitly that the variable obtained after mutating at the vertex corresponding to the underlined $\underline{i}$ is regular.

Exercise 10.3. Suppose $|i - j| = 1$. Suppose also that $\mathbf{i}$ and $\mathbf{i}'$ are braid letters of the following form

$$\mathbf{i} = \mathbf{j_1}\underline{i}ji\mathbf{j}_2, \qquad \mathbf{i}' = \mathbf{j}_1\underline{j}ij\mathbf{j_2}.$$

Show that $Q_{\mathbf{i}}$ and $Q_{\mathbf{i}'}$ differ by a mutation, precisely at the vertex corresponding to the underlined letters above.

Exercise 10.4. Consider the braid word $\mathbf{i} = 1^n$, that is, $\mathbf{i} = \underbrace{11\cdots1}_{n \text{ times}}$. Use the Starfish lemma and Exercise 9.2 above to show that $(x_{\mathbf{i}}, Q_{\mathbf{i}})$ indeed endows $\mathbb{C}[\mathrm{BS}(\mathbf{i})]$ with a cluster structure.

Exercise 10.5. Recall that $\mathbb{C}[\mathrm{BS}(\mathbf{i})] \cong \mathbb{C}[z_1, \ldots, z_r][f_1^{-1}, \ldots, f_{k-1}^{-1}]$, where $f_1, \ldots, f_k$ are the principal minors of the matrix $B_{\mathbf{i}}(z_1, \ldots, z_r)$. Consider the cluster structure on $\mathbb{C}[\mathrm{BS}(\mathbf{i})]$ constructed in the lecture. Show that all cluster variables (not just those in the initial seed) in $\mathbb{C}[\mathrm{BS}(\mathbf{i})]$ belong to $\mathbb{C}[z_1, \ldots, z_r]$, that is, they do not involve negative powers of frozen variables.

Exercise 10.6. Let $\tilde{T} \subseteq \mathrm{GL}(k)$ be the torus of diagonal matrices, and let T be the quotient of $\tilde{T}$ by the subgroup of scalar matrices (diagonal matrices where all diagonal entries are equal). Note that the torus $\tilde{T}$ acts on the flag variety, and this action factors through its quotient T.

(a) Show that T preserves relative position, i.e., if $\mathcal{F}^1_\bullet \xrightarrow{w} \mathcal{F}^2_\bullet$ and $t \in T$, then $t.\mathcal{F}^1_\bullet \xrightarrow{w} t.\mathcal{F}^2_\bullet$. Conclude that T acts naturally on $\mathrm{BS}(\mathbf{i})$, for any braid word $\mathbf{i}$ in the letters $\{1, \ldots, k-1\}$.

(b) Let $\mathbf{i} = (i_1, \ldots, i_r)$ be a braid word such that $s_{i_1} \cdots s_{i_r} \in S_k$ is a single k-cycle. Show that T acts freely on $\mathrm{BS}(\mathbf{i})$. For this, it may be useful to explicitly give the action of T on coordinates $(z_1, \ldots, z_r)$.

In fact, the converse of (b) is also true: T acts freely on $\mathrm{BS}(i_1, \ldots, i_r)$ if and only if $s_{i_1} \cdots s_{i_r}$ is a single k-cycle.

Note that by the same reasoning, one gets an action of T on the braid variety $X(\beta)$ for any braid β. But the converse of (b) is not valid for general braid varieties – it is one of the many results that are valid for double Bott-Samelson varieties, but not general braid varieties.

Exercise 10.7. Let $k < n$, and consider the variety of n-tuples of points $(v_1, \ldots, v_n) \in (\mathbb{C}^k)^n$ such that $v_i, v_{i+1}, \ldots, v_{i+k-1}$ are linearly independent for every $i = 1, \ldots, n$, where the indices are taken mod n. Let $\Pi^\circ_{k,n}$ be the subvariety consisting of those tuples $(v_1, \ldots, v_n)$ such that $v_{n-k+1} = e_1, v_{n-k+2} = e_2, \ldots, v_n = e_k$. (If you want, you can assume $k = 4, n = 8$ – once you have cracked this case the general pattern should be clear)

(a) Consider the k-stranded braid $\beta_{k,n} = (\sigma_{k-1} \cdots \sigma_1)^{n-k}$. Exhibit a map $\Pi^\circ_{k,n} \to \mathrm{BS}(\beta_{k,n})$.

(b) Show that $\Pi^\circ_{k,n}$ is isomorphic to the open positroid variety Π^e_w, where $e \in S_n$ is the identity and $w = (n-k+1) \cdots n12 \cdots (n-k)$.

Note that, by dimension reasons, the map you constructed in a) cannot be an isomorphism – the positroid variety has dimension $k(n-k)$ while the double Bott-Samelson variety $\mathrm{BS}(\beta_{k,n})$ has dimension $(k-1)(n-k)$. In fact, $\Pi^\circ_{k,n} \cong \mathrm{BS}(\beta_{k,n}) \times (\mathbb{C}^\times)^{n-k}$, a fact that can be explained using cluster algebras as follows.

(c) Go to Page 25 in `https://arxiv.org/pdf/2008.09189` and see the quiver and cluster variables there for the case $k = 3, n = 7$ (for example, the variable (457) should be interpreted as being the determinant of $\begin{bmatrix} v_4 \\ v_5 \\ v_7 \end{bmatrix}$). Setting the upper left corner equal to 1, we obtain a cluster structure on $\Pi^\circ_{3,7}$. Compare this quiver with the one you obtain from $\mathrm{BS}(\beta_{3,7})$. How are they different? Now (the proof of) Proposition 5.11 in `https://arxiv.org/pdf/1604.06843` implies the desired result (granted, this proposition is not easy to read!)