Lecture 1: main exercises

Exercise 1.1. Consider a matrix $M \in \mathrm{GL}(n)$ and let $i = 1, \dots, n-1$.

(a) Show that $\mathcal{F}_\bullet(M) \xrightarrow{s_i} \mathcal{F}'_\bullet$ if and only if there exists a unique $z \in \mathbb{C}$ such that

$$\mathcal{F}'_\bullet = \mathcal{F}_\bullet(MB_i(z)), \qquad B_i(z) = \begin{pmatrix} 1 & \cdots & & & \cdots & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ 0 & \cdots & z & -1 & \cdots & 0 \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 1 \end{pmatrix}$$

where the non-identity part of $B_i(z)$ is located at the i-th and $(i+1)$-st row and columns.

(b) Thanks to part (a), if we fix a flag $\mathcal{F}_\bullet$ we have $\{\mathcal{F}'_\bullet \mid \mathcal{F}_\bullet \xrightarrow{s_i} \mathcal{F}'_\bullet\} \cong \mathbb{C}$. However, this parametrization depends on the matrix chosen to represent $\mathcal{F}_\bullet$: if $\mathcal{F}_\bullet(M) = \mathcal{F}_\bullet(M')$ then for every $z \in \mathbb{C}$ there exists $w \in \mathbb{C}$ such that $\mathcal{F}_\bullet(MB_i(z)) = \mathcal{F}_\bullet(M'B_i(w))$. How are M and M' related? How are z and w related?

Exercise 1.2. (a) Let $w \in S_n$, assume $w(i) < w(i+1)$, so $w < ws_i$ in Bruhat order. Show that $(BwB)(Bs_iB) = Bws_iB$. Hint: What does a general matrix in wBs_i look like? It may be helpful to first think what a matrix in wB looks like

(b) Under the assumptions of (a), conclude that:

- If $\mathcal{F}^1_\bullet, \mathcal{F}^2_\bullet, \mathcal{F}^3_\bullet$ are flags such that $\mathcal{F}^1_\bullet \xrightarrow{w} \mathcal{F}^2_\bullet$ and $\mathcal{F}^2_\bullet \xrightarrow{s_i} \mathcal{F}^3_\bullet$, then $\mathcal{F}^1_\bullet \xrightarrow{ws_i} \mathcal{F}^3_\bullet$.
- If $\mathcal{F}^1_\bullet \xrightarrow{ws_i} \mathcal{F}^3_\bullet$, then there exists a unique flag $\mathcal{F}^2_\bullet$ such that $\mathcal{F}^1_\bullet \xrightarrow{w} \mathcal{F}^2_\bullet$ and $\mathcal{F}^2_\bullet \xrightarrow{s_i} \mathcal{F}^3_\bullet$.

Exercise 1.3. (a) Assume that we have $\mathcal{F}^1_\bullet \xrightarrow{s_i} \mathcal{F}^2_\bullet \xrightarrow{s_j} \mathcal{F}^3_\bullet \xrightarrow{s_i} \mathcal{F}^4_\bullet$, where $|i-j| = 1$. Show that $\mathcal{F}^2_\bullet$ and $\mathcal{F}^3_\bullet$ are uniquely determined by $\mathcal{F}^1_\bullet$ and $\mathcal{F}^4_\bullet$.

(b) Conclude from a) that, if $\mathbf{i} = \mathbf{i}_1 iji \mathbf{i}_2$ and $\mathbf{j} = \mathbf{i}_1 jij \mathbf{i}_2$ then the varieties $X(\mathbf{i})$ and $X(\mathbf{j})$ are isomorphic.

(c) Assume that we have $\mathcal{F}^1_\bullet \xrightarrow{s_i} \mathcal{F}^2_\bullet \xrightarrow{s_j} \mathcal{F}^3_\bullet$, where $|i-j| > 1$. Show that $\mathcal{F}^2_\bullet$ is uniquely determined by $\mathcal{F}^1_\bullet$ and $\mathcal{F}^3_\bullet$.

(d) Conclude from c) that if $\mathbf{i} = \mathbf{i}_1 ij \mathbf{i}_2$ and $\mathbf{j} = \mathbf{i}_1 ji \mathbf{i}_2$ then the varieties $X(\mathbf{i})$ and $X(\mathbf{j})$ are isomorphic.

Thanks to this, if $\mathbf{i}$ and $\mathbf{j}$ are words related by braid moves, then we have a canonical isomorphism $X(\mathbf{i}) \cong X(\mathbf{j})$. So $X(\mathbf{i})$ depends only on the braid $\beta = \sigma_{i_1} \cdots \sigma_{i_r} \in \mathrm{Br}_n$ defined by $\mathbf{i}$, and not on $\mathbf{i}$ itself. Hence the name *braid variety*.

Lecture 1: additional exercises

Exercise 1.4. Let $\mathcal{F}^1_\bullet, \mathcal{F}^2_\bullet$ be flags in $\mathbb{C}^n$. In this exercise we will verify that there is a unique permutation w such that $\mathcal{F}^1_\bullet \xrightarrow{w} \mathcal{F}^2_\bullet$.

(a) Let $W_{ij} := (\mathcal{F}^1_i \cap \mathcal{F}^2_j)/(\mathcal{F}^1_i \cap \mathcal{F}^2_{j-1})$. Show that $\dim W_{ij}$ is 0 or 1.

(b) Let $w(j)$ be the minimum value of i such that $\dim W_{ij} = 1$. Show that w is a permutation.

(c) Similarly, let $W'_{ij} = (\mathcal{F}^1_i \cap \mathcal{F}^2_j)/(\mathcal{F}^1_{i-1} \cap \mathcal{F}^2_j)$, and let $w'(i)$ be the minimum j such that $\dim W'_{ij} = 1$. Show that w' is the inverse of w.

(d) Let e_i be a vector in $\mathcal{F}^1_i \cap \mathcal{F}^2_{w'(i)}$ that is *not* in $\mathcal{F}^1_{i-1}$. Show that $(e_1, \dots, e_i)$ is a basis of $\mathcal{F}^1_i$ and $(e_{w(1)}, \dots, e_{w(i)})$ is a basis of $\mathcal{F}^2_i$ for every i. Conclude that $\mathcal{F}^1_\bullet \xrightarrow{w} \mathcal{F}^2_\bullet$.