Math 607 (Homological Algebra), Spring 2016
Exercises

I do not expect people to do all the exercises from week to week, but perhaps you should at some point in your life! Instead, try to do the computational ones and some of the abstract ones, to get a handle on the material. I have marked the most essential problems with (@).

Starred problems take more work or are harder than unstarred problems, but are still worth doing. Double starred problems are even harder. Triple starred problems I'm not even sure I could do.

Stuff you should have read so far:

1. MacLane Ch 8, on abelian categories.
2. Weibel Ch 1.1, 1.3, 1.4: basics on complexes, homology, and the long exact sequence.
3. Weibel Ch 1.6, the Freyd-Mitchell embedding theorem
4. Weibel Ch 2.1-2.5, basics of derived functors
5. Weibel Ch 2.6, MacLane Ch 4.1: adjoint functors, units and counits, exactness properties. Tensor-Hom adjunction.
6. (Looking for a good source) Frobenius algebras
7. Weibel 3.4, Eisenbud p652-656: Yoneda ext.
8. Weibel Lemma 3.2.8 for F-acyclic resolutions
9. Weibel Ch 4.1, 4.2, 4.5: homological dimension and Koszul complexes. Chapter 1 of Crawley-Boevey, lectures on representations of quivers. Wikipedia for bar resolution.

Week 1 exercises
Baby representation theory

1. Let x be an operator acting on a finite dimensional vector space V by a single Jordan block, with eigenvalue λ. Show that $\mathrm{Ker}(x - \lambda I)$ is contained in any nonzero subspace.
2. Let $\mathcal{C}$ be the category of $\mathbb{R}[x]$ modules which are finite dimensional vector spaces over $\mathbb{R}$. What are the indecomposable objects in $\mathcal{C}$? (That is, what is real Jordan normal form? Try to derive it if you don't know it.)
3. Consider a short exact sequence $\left(0 \to B \xrightarrow{f} M \xrightarrow{g} A \to 0\right)$ of R-modules for some ring R.
   (a) (@) Prove that the sequence is trivial (i.e. split) if and only if there exists $M \xrightarrow{p} B$ such that $pf = \mathrm{id}_B$, if and only if there exists $A \xrightarrow{q} M$ such that $gq = \mathrm{id}_A$. (Hint: Consider the images of f and q, or the kernels of p and g.) (Aside: your argument is probably element-theoretical, that is, it uses the underlying sets of A, B, and M. Eventually we'll see an abstract proof that works in any abelian category, where objects don't necessarily have elements.)

(b) (**) Prove that the sequence is trivial if and only if it is apparently split (i.e. $M \cong A \oplus B$) in the case when A, B, and M are finitely generated abelian groups.

4. (@) Find all simples and indecomposables (up to isomorphism) in R-mod, for the ring $R = \mathbb{C}[x]/x^4$. Find all non-trivial extensions between indecomposables. Find all indecomposable projectives.

5. Find all simples and indecomposables (up to isomorphism) in Q-rep, for the quivers below. Find all non-trivial extensions between indecomposables. Find all indecomposable projectives.

   (a) (@) $\bullet \to \bullet \to \bullet$

   (b) $\bullet \to \bullet \leftarrow \bullet$ (two outer vertices with an edge to the central vertex)

   (c) (*) Like the previous example, but three outer vertices each with an edge to the center vertex.

   (d) (*) This time there are four outer vertices with edges to the central vertex. But don't try to find all the indecomposables (there are infinitely many!). Try to find an indecomposable where the dimension of the vector space on the outer vertices is 2, and the dimension on the central vertex is 4.

   (e) (**) For the four outer vertex case, construct infinitely many non-isomorphic indecomposable representations.

6. (@) Let $\mathcal{C}$ be the category of complexes which are finite dimensional vector spaces in each homological degree. Find all simples and indecomposables in $\mathcal{C}$. Find all non-trivial extensions and indecomposable projectives.

7. (*) (@) Let $\mathcal{C}$ be the category of bounded complexes which are finitely generated FREE abelian groups in each homological degree. This is an additive category, not an abelian one. Find all indecomposables in $\mathcal{C}$. (Note: you should look up some facts about homomorphisms between free abelian groups, namely, invariant factors and Smith normal form.)

8. (*) Let $\mathcal{C}$ be the category of bounded bicomplexes of finite dimensional vector spaces. That is, one has a vector space $V_{i,j}$ for each i and j in $\mathbb{Z}$, which are zero for $|i|+|j| >> 0$. One also has differentials $d_1 \colon V_{i,j} \to V_{i+1,j}$ and $d_2 \colon V_{i,j} \to V_{i,j+1}$ such that $d_1^2 = 0 = d_2^2$ and $d_1 d_2 = -d_2 d_1$. Find all simples and indecomposables in $\mathcal{C}$.

Categorical nonsense and abelian categories

1. Prove that, in an $\mathcal{A}$b-category, (finite) products are coproducts are direct sums. Show that infinite products and infinite coproducts need not agree.

2. (@) Let $F \colon \mathcal{A} \to \mathcal{B}$ be a functor between $\mathcal{A}$b-categories, such that $\mathcal{A}$ has finite direct sums. Show that F preserves addition (i.e. $F(f+g) = F(f) + F(g)$ for all parallel morphisms f and g) if and only if F preserves direct sums (i.e. F sends the structure maps of a direct sum into the structure maps of a direct sum). Hint: describe addition using various canonical maps, such as the diagonal map $M \to M \oplus M$, etcetera.

3. Continuing the previous example, do exercise 4 from p198 of MacLane.

4. In an additive category, prove that an equalizer of parallel morphisms f and g, if it exists, is also an equalizer of $f-h$ and $g-h$ for any parallel morphism h.

5. (@) Prove the following statements in an additive category.

   (a) $M \cong 0$ if and only if $\mathrm{id}_M = 0$.

   (b) f is monic if and only if $\operatorname{Ker} f$ exists and $\operatorname{Ker} f \cong 0$.

   (c) $\operatorname{Ker}(M \to 0)$ exists and $M \overset{\mathrm{id}_M}{\to} M$ is such a kernel.

   (d) Suppose $\operatorname{Ker}(f)$ exists. Then a map $\pi\colon L \to M$ lifts to a map $\tilde{\pi}\colon L \to \operatorname{Ker}(f)$ with $\ker(f) \circ \tilde{\pi} = \pi$ if and only if $f\pi = 0$. Moreover, in this case $\tilde{\pi}$ is unique.

6. In a preabelian category, prove that the following statements are equivalent.

   - Every monic morphism is a kernel, and every epic morphism is a cokernel.
   - For every morphism f, the canonical map $\iota\colon \operatorname{Coker}(\ker(f)) \to \operatorname{Ker}(\operatorname{coker}(f))$ is an isomorphism.

7. (@) Confirm that $\operatorname{Rep} Q$ is an abelian category, for any quiver Q.

An extended example to help understand blocks and generalized eigenvalues
These next problems are all about the ring $A = \mathbb{C}[x,h]/(hx - xh - 2x)$.

1. We explore some full subcategories $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$ of A-mod.

   (a) (@) Suppose that V is an A-module and $v \in V$ satisfies $hv = \lambda v$ for some $\lambda \in \mathbb{C}$. Show that xv is also an eigenvector for h, with eigenvalue $\lambda + 2$.

   (b) (@) Let $\mathcal{A}$ denote the full subcategory of A-modules for which h acts diagonalizably. That is, V is an object of $\mathcal{A}$ (we just write $V \in \mathcal{A}$) if it has a (possibly infinite) $\mathbb{C}$-basis consisting of h-eigenvectors. Show that $\mathcal{A}$ is an abelian category. Is it closed under extension inside A-modules?

   (c) (@) Let $\mathcal{B}$ denote the full subcategory of A-modules for which h acts "Jordan-ly." That is, $V \in \mathcal{B}$ if it has a $\mathbb{C}$-basis which partitions into disjoint sets on which h acts like a Jordan block. Show that $\mathcal{B}$ is an abelian category. Is it closed under extension inside A-modules? Find an object in $\mathcal{B}$ but not in $\mathcal{A}$.

   (d) (@) Show that for any $V \in \mathcal{B}$, the set $\{\mathrm{Re}\lambda \mid \lambda \text{ is an eigenvalue of } h\} \subset \mathbb{R}$ is bounded below. (Modules in $\mathcal{B}$ are assumed to be finitely generated).

   (e) Let $\mathcal{C}$ denote the full subcategory of A-Modules for which h acts locally finitely. That is, $V \in \mathcal{C}$ if, for any $v \in V$, there is a finite dimensional subspace $W \subset V$ with $v \in W$, which is preserved under the action of h (but not necessarily of x). Show that $\mathcal{C}$ is an abelian category. Is it closed under extension inside A-Modules? Find an object in $\mathcal{C}$ which is not in $\mathcal{B}$. (There are many such examples, like an infinite direct sum of objects in $\mathcal{B}$. Find a counterexample that illustrates the difference between acting locally finitely and Jordan-ly.

   (f) Show that any finitely generated module in $\mathcal{C}$ is actually in $\mathcal{B}$, and that each generalized eigenspace of h is finite-dimensional.

2. (@) In this problem, we study $\mathcal{A}$.

(a) Given $M \in \mathcal{A}$, let M_λ denote the λ-eigenspace of h inside M. Show that the functor $F_\lambda \colon \mathcal{A} \to \mathrm{vect}$ sending $M \mapsto M_\lambda$ is, indeed, a functor. Moreover, show this functor is exact.

(b) Show that F_λ is representable, and find the object P_λ such that $F_\lambda = \mathrm{Hom}(P_\lambda, \bullet)$.

(c) Is P_λ projective in $\mathcal{A}$? Is P_λ projective in $\mathcal{B}$?

(d) Classify the indecomposable objects in $\mathcal{A}$, and prove that each has a two-term projective resolution.

3. In this problem, we discuss blocks of $\mathcal{A}$.

   (a) Let $[\lambda]$ denote the set $\{\lambda + 2n \mid n \in \mathbb{Z}\} \subset \mathbb{C}$. Let $\mathcal{A}_{[\lambda]}$ denote the full subcategory consisting of modules M for which $M_\mu = 0$ whenever $\mu \notin [\lambda]$. Show that $\mathcal{A}_{[\lambda]}$ is abelian, and closed under extension in $\mathcal{A}$.

   (b) Show that there are no nontrivial extensions between an object in $\mathcal{A}_{[\lambda]}$ and an object in $\mathcal{A}_{[\mu]}$ when $[\lambda] \neq [\mu]$.

   (c) Show that $\mathcal{A} \cong \oplus_{[\lambda]} \mathcal{A}_{[\lambda]}$. That is, for any $M \in \mathcal{A}$, there is a canonical (i.e. functorial) projection $\pi_{[\lambda]} \colon M \to \pi_{[\lambda]} M$, where $\pi_{[\lambda]} M \in \mathcal{A}_{[\lambda]}$, and $M \cong \oplus_{[\lambda]} \pi_{[\lambda]} M$. These subcategories $\mathcal{A}_{[\lambda]}$ are often called *blocks* because they do not interact with each other.

These are a few of my favorite rings

For each of these categories, the default question is: find projective resolutions of each indecomposable object. Then, for extra fun: Make these projective resolutions as short as possible: what is this shortest length? If the length is infinite, can you make the resolution periodic?

1. (@) $\mathbb{Z}$-mod.

2. (@) $\mathbb{C}[x]/x^m$-mod.

3. (@) $\mathbb{C}[x]$-mod.

4. $\mathbb{C}[x, y]$-mod, just do the irreducible modules. Continue with $\mathbb{C}[x, y, z]$ if you don't get it yet.

5. (*) Let $R = \mathbb{C}[x, y, A, B]/(xA + yB)$. Find a projective resolution of $M = R/(x, y)$.

6. (@) The category of quiver representations for $\bullet \to \bullet \to \bullet$

7. Quiver reps for $\bullet \to \bullet \leftarrow \bullet$.

8. (**) Quiver reps for an arbitrary finite quiver without loops. (This will be very hard unless you look it up, because the easy proofs use technology from later in the class. However, maybe you had fun and saw a pattern and are super clever.)

9. In this problem, let $R = \mathbb{Z}[x]/(x^2 - 1)$. Note that this is just the group algebra of $\mathbb{Z}/2\mathbb{Z}$, but since we are not over a field (and could specialize to characteristic 2) it is not semisimple. Let $T = \mathbb{Z}$ be the R-module where x acts as 1, and let $S = \mathbb{Z}$ be the R-module where x acts as -1. Show that S (resp. T) is both a submodule and a quotient of R, but is nonetheless not a summand. Find projective resolutions of S and T. We will return to this ring later in life.

Projective resolutions

1. (a) (@) Let $\mathcal{A}$ be an abelian category. Prove that $\mathcal{C}\mathrm{h}(\mathcal{A})$ is an abelian category, as are its bounded-above, bounded-below, and bounded versions.
   (b) (@) Prove that kernels and cokernels in $\mathcal{C}\mathrm{h}(\mathcal{A})$ behave as expected, and that a sequence of complexes is exact if and only if it is exact in each homological degree.
2. (@) Fix an abelian category $\mathcal{A}$ with enough projectives. Let $K^\bullet \in \mathcal{C}\mathrm{h}(\mathcal{A})$ be a complex which is bounded above. A *projective resolution* of $K^\bullet$ is a complex $P^\bullet$ of projective objects, together with a surjection $\pi\colon P^\bullet \to K^\bullet$ such that π is a quasi-isomorphism. Prove that every bounded above complex has a projective resolution.
3. (*) Prove that any quasi-isomorphism between bounded-above complexes of projectives is a homotopy equivalence. (You can do a direct proof now, although there will be a much slicker proof later in the course.)
4. A complex $C^\bullet$ is *split* if there are maps $s\colon C^n \to C^{n-1}$ such that $d = dsd$. It is *split exact* if it is both split and exact.
   (a) (@) Prove that $C^\bullet$ is split exact if and only if the identity map is nulhomotopic. We also call C *contractible*.
   (b) (@) Find an exact complex of abelian groups that is not split exact.
   (c) Prove that an exact complex of free abelian groups is split exact.
   (d) Prove that an exact complex of projective objects which is bounded is split exact. What about bounded above? Bounded below?
   (e) (*) Prove that a complex is projective (in the category of complexes) if and only if it is a split exact complex where each term is projective.
5. (@) Dimension shifting: We work in $\mathcal{A}$, an abelian category with enough projectives, and let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor. Suppose that $0 \to K \to P_m \to P_{m-1} \to \cdots \to P_0 \to M \to 0$ is exact in $\mathcal{A}$, with P_i projective. Show that $L^{-i}F(M) \cong L^{-i+m+1}F(K)$ for all $i \geq m+2$, and that $L^{-m-1}F(M) \to F(K)$ is injective, being the kernel of $F(K) \to F(P_m)$.

Computing Ext and Tor
We return to some of my favorite rings.

1. (@) Recall the ring $R = \mathbb{Z}[x]/(x^2-1)$ and its modules S and T from a previous exercise. Compute $\mathrm{Ext}^i(S,S)$, $\mathrm{Ext}^i(S,T)$, $\mathrm{Ext}^i(T,S)$, and $\mathrm{Ext}^i(T,T)$. Compute $\mathrm{Tor}^i(S,S)$, $\mathrm{Tor}^i(S,T)$ and $\mathrm{Tor}^i(T,S)$.
2. (@) Compute $\mathrm{Ext}^i(L,M)$ for all simple modules L, M of $\mathbb{C}[x]$.
3. Repeat the previous problem for $\mathbb{C}[x,y]$ (it should be equally easy, if you found the projective resolutions before).
4. (@) Compute Ext^i between simple objects in the category of quiver representations for one of the three-vertex quivers above.

5. Compute all Ext and Tor between indecomposable modules for $\mathbb{C}[x]/x^4$.

Adjunctions and Frobenius extensions

1. (@) Prove the Yoneda Exactness Lemma: given $A \xrightarrow{f} B \xrightarrow{g} C$ which becomes exact after applying $\mathrm{Hom}(\bullet, M)$ for any M, then it was exact to begin with. Find an EXPLICIT example where $A \xrightarrow{f} B \xrightarrow{g} C$ is exact, but is not exact after applying $\mathrm{Hom}(\bullet, M)$ for some M.

2. (a) Prove that a finite-dimensional algebra A is Frobenius if and only if it has a non-degenerate associative bilinear form.
   (b) Prove that $n \times n$ matrices are a Frobenius algebra, and more generally, that any semisimple algebra over $\mathbb{C}$ is Frobenius.
   (c) Prove that any two Frobenius structures on a Frobenius algebra are equivalent under the action of a unit in A.
   (d) Verify that a Frobenius algebra is a coalgebra, and that comultiplication is an A-bimodule map.
   (e) Compute the Nakayama automorphism for the exterior algebra in n variables.

3. Find the unit and counit of adjunction explicitly, for the following adjunctions:
   (a) (@) induction-restriction.
   (b) tensor-hom
   (c) restriction-induction for a Frobenius extension.

4. (@) Show that $\mathbb{C}[x^2] \subset \mathbb{C}[x]$ is a Frobenius extension. Compute the Frobenius trace map, and the coproduct.

Barr-Beck

In this long and unnecessary exercise, we explore the Barr-Beck theorem. For certain undefined terms see Ch 6 of MacLane.

We know a plethora of examples of categories which are sets with additional structure. These categories typically have a forgetful functor to sets, and an adjoint functor back. Let $F\colon \mathcal{A} \to \mathcal{B}$ be left adjoint to $G\colon \mathcal{B} \to \mathcal{A}$, where G is thought of as the forgetful functor. The Barr-Beck theorem tells you when an object in $\mathcal{B}$ can be thought of as an object in $\mathcal{A}$ equipped with some additional structure, and precisely what that additional structure is, solely using the properties of F and G. So, for instance, one can recover the definition of a group (i.e. a set G with certain structure maps satisfying certain axioms) without needing to know it beforehand, if one knows the category of groups and the free-forget adjoint pair.

However, the Barr-Beck theorem does not always apply! It holds when G satisfies some axioms, which are somewhat thorny. Here is a slightly weaker version.

Definition 0.1. A functor $G\colon \mathcal{B} \to \mathcal{A}$ is *conservative* if $G\varphi$ is an isomorphism if and only if φ is an isomorphism.

Theorem 0.2. Suppose that $\mathcal{B}$ and $\mathcal{A}$ have all colimits, and F and G are an adjoint pair as above. Suppose that G is conservative and preserves colimits. Then G is monadic. This means that $\mathcal{B}$ is equivalent to the category of modules in $\mathcal{A}$ over the algebra GF (to be explained below).

1. For which of the following categories is the forgetful functor to sets conservative: groups, topological spaces, R-modules?

2. What about the restriction functor from S-modules to R-modules, when $R \subset S$ is a ring extension?

3. In an abelian category $\mathcal{B}$ which is linear over a field $\Bbbk$, one has a functor $\mathrm{Hom}(P, \bullet) \colon \mathcal{B} \to \mathrm{Vect}$ for any object P. Show that this functor is conservative if and only if P is a generator. A *generator* is an object P for which $\mathrm{Hom}(P, M) = 0$ implies that $M = 0$.

4. Prove that a right adjoint preserves limits and a left adjoint preserves colimits. Show that a right adjoint need not preserve colimits.

5. Show that an additive functor between abelian categories preserves colimits if and only if it preserves coequalizers.

6. Show that $\mathrm{Hom}(P, \bullet)$ preserves colimits if and only if P is projective.

7. Consider $A = GF$, an endofunctor of $\mathcal{A}$. Using the units and counits of adjunction, give A the structure of an algebra in the category of endofunctors of $\mathcal{A}$. That is, construct a multiplication map $A \circ A \to A$, and a unit map $\mathbb{1} \to A$, and check associativity and the unit axiom. (An algebra in a category of endofunctors is sometimes called a *monad*.)

8. Describe this algebra explicitly in the following three settings:

   (a) G is the forgetful map from (nonunital) monoids to sets, and F is the free monoid (whose elements are words in the given set).

   (b) G is $\mathrm{Hom}(P, \bullet) \colon \mathcal{B} \to \mathrm{Vect}$ for an object in $\mathcal{B}$, and F sends the one-dimensional vector space to P.

   (c) G is the restriction functor from S-modules to R-modules, for $R \subset S$.

9. One defines an A-module in $\mathcal{A}$ to be an object M equipped with a map $AM \to M$, satisfying associativity and the unit axiom. (An A-module is sometimes, poorly in my opinion, called an *algebra over the monad*.) These form a category, with morphisms being module maps. Describe what it means to be an A-module in the three examples above.

10. In the first example, why is an A-module in sets the same thing as a monoid? How are the definitions different?

11. Show that $G(X)$ is naturally an A-module, for any object $X \in \mathcal{B}$.

12. Deduce from the Barr-Beck theorem that if P is a projective generator of $\mathcal{B}$, then $\mathcal{B}$ is equivalent to the category of modules over $\mathrm{End}(P)^{\mathrm{op}}$. This is the main theorem of Morita theory.

Yoneda Ext

1. (@) As a warmup, prove that pullbacks preserve kernels and pushforwards preserve cokernels.

2. (@) Suppose that P is projective, and $(SES1) = (0 \to M \to P \to A \to 0)$ and $(SES2) = (0 \to B \to X \to A \to 0)$ are exact. Prove that, under the dimension-shifting isomorphism $\mathrm{Ext}^1(A, B) \cong \mathrm{Hom}(M, B)$, the element $\delta(\mathrm{id}_A) \in \mathrm{Ext}^1(A, B)$ coming from the long exact sequence attached to SES2 agrees is sent to the map $M \to B$ in any map from SES1 to SES2 lifting the identity map of A.

3. (@) Compute $\mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ in the category of $\mathbb{Z}$-modules, and describe the corresponding extensions for each element.

4. Prove that, for $k > 0$ an element of $\mathrm{Ext}^k(A, B)$ is zero if and only if the corresponding long extension $0 \to B \to X_k \to \ldots \to X_1 \to X_0 \to A$ is equivalent to one which splits into a direct sum of two complexes, one containing A (and mapping in via id_A) and the other containing B (and mapping in via id_B).

5. Let $\mathbb{C}$ denote the $\mathbb{C}[x]$-module on which x acts by zero. In this exercise we explore $\mathrm{Ext}^*(\mathbb{C}, \mathbb{C})$ in various categories: $\mathbb{C}[x]$-modules, $\mathbb{C}[x]/x^2$-modules, and $\mathbb{C}[x]/x^d$-modules for $d > 2$.

   (a) (@) Compute $\mathrm{Ext}^*(\mathbb{C}, \mathbb{C})$ as a vector space in all three settings.

   (b) (@) Let $y \in \mathrm{Ext}^1(\mathbb{C}, \mathbb{C})$ denote the extension $0 \to \mathbb{C} \to \mathbb{C}[x]/x^2 \to \mathbb{C} \to 0$. Write down the long extension corresponding to y^k (it looks the same in all the categories above).

   (c) (@) Prove that y^2 is nonzero for $\mathbb{C}[x]/x^2$-modules. Prove that $y^2 = 0$ for $\mathbb{C}[x]/x^d$-module for any $d > 2$. Prove that $y^2 = 0$ for $\mathbb{C}[x]$-modules. Do this all using the previous exercise, by finding a length 2 long extension which is split that maps to it, or proving that none exists.

   (d) For $\mathbb{C}[x]/x^d$-modules with $d > 2$, find a non-trivial element of $\mathrm{Ext}^2(\mathbb{C}, \mathbb{C})$. Call this element z.

   (e) (*) Prove that $\mathrm{Ext}^*(\mathbb{C}, \mathbb{C})$ in the category of $\mathbb{C}[x]/x^d$-modules for $d > 2$ is isomorphic as an algebra to $\mathbb{C}[y, z]/y^2$. This is unlike the $d = 2$ case, where $\mathrm{Ext}^*(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}[y]$.

Acyclic resolutions
I could not think of many good exercises here. But we do have something I used in class.

1. (@) Suppose that $S^\bullet$ is an exact bounded above complex of flat right R-modules. Prove that for any left R-module M, the complex $S^\bullet \otimes_R M$ is exact.

Homological Dimension

1. Baer's criterion:

   (a) (@) Do Weibel exercise 2.3.1. Then read Example 2.3.3. Is $\mathbb{Z}[\frac{1}{p}]$ divisible? Why is $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ divisible?

   (b) Prove that $\mathbb{Q}/\mathbb{Z}$ splits as a direct sum, over all p, of $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$.

   (c) Prove that the global dimension of a ring R is the supremum of $\mathrm{pd}(R/I)$ for all ideals I.

2. Given an s.e.s. $0 \to A \to B \to C \to 0$, prove that $\mathrm{pd}(B) \leq \max\{\mathrm{pd}(A), \mathrm{pd}(C)\}$, with equality unless $\mathrm{pd}(C) = \mathrm{pd}(A) + 1$.

3. Prove that $\Bbbk[G]$, the group algebra of a finite group G, has infinite homological dimension when the characteristic of $\Bbbk$ divides G. Prove that $\mathbb{Z}[G]$ has infinite homological dimension. C.f. the last problem of "a few of my favorite rings."

4. Let R be a commutative domain, with fraction field F.

   (a) Let $I^{-} \subset F$ denote the set $\{x \in F \mid xy \in R \text{ for all } y \in I\}$. Clearly I^{-} is an R-submodule of F. Show that $I^{-} \cong \mathrm{Hom}(I, R)$.

   (b) We let II^{-} denote the span of $\{xy \in F \mid x \in I, y \in I^{-}\}$. Prove that $II^{-} = R$ (we call I *invertible*) if and only if I is projective as an R-module. (Hint: the statement $II^{-} = R$ is equivalent to $1 \in II^{-}$. Choose an expression of 1 as a linear combination, and use this to produce a split inclusion of I into R^m, where there are m terms in this linear combination.)

   (c) Let $I = (x, y) \subset \mathbb{C}[x, y]$. Is I invertible? Deduce that $\mathbb{C}[x, y]$ is not hereditary.

5. Check that the bimodule bar resolution is an exact complex of bimodules.

6. (@) Find the canonical projective resolutions of all the simple representations for the A_3 and D_4 quivers, with the orientation where the central vertex is a sink. (Then do it for the rest of the indecomposable representations too.)

7. (@) Let $R = \oplus_{i \in \mathbb{Z}} R^i$ be a graded commutative $\Bbbk$-algebra (for a field $\Bbbk$), where $R^i = 0$ for $i < 0$, and $R^0 = \Bbbk$. Assume that R is finite dimensional; in particular, it is bounded above. Prove that R has infinite global dimension. (Hint: Use the fact that projective modules over a local ring are free.)