Math 432/532 (Topology), Winter 2016
Take Home Final

Starred problems are for 531 students, and are extra credit for 431 students. Note to 431 students: starred problems are not worth as much as normal problems, so budget your time accordingly.

You may use results from class and HW, provided you cite them correctly. You may consult G+P, Kosniowski, Munkres, and Wikipedia. Other websites must be cited, and you may not use answer guides or question answering forums... you get the idea. You may not consult other people.

1. Let $X_0 = \{(x, y, z) \in \mathbb{R}^3 \mid 2x^2 + 5y^2 - z^2 - yz - 12x = 0\}$.
   (a) Show that X_0 is a 2-manifold.
   (b) Find a basis for T_pX_0 at the point $p = (12, 0, 12)$.
   (c) At which points is $\pi\colon X_0 \to \mathbb{R}^2$, $(x, y, z) \mapsto (y, z)$ a local diffeomorphism?
   (d) (*) At which points is $g\colon X_0 \to \mathbb{R}$, $(x, y, z) \mapsto x$ a submersion?
   (e) Prove that, for almost every $a \in \mathbb{R}$, the set $X_a = \{(x, y, z) \in \mathbb{R}^3 \mid 2x^2 + 5y^2 - z^2 - yz - 12x = a\}$ is a 2-manifold.

2. Let $Z = \{[x : y : z : w] \in \mathbb{RP}^3 \mid xy = z^2 + w^2\}$. Let U_x (resp. U_y) denote the open set in $\mathbb{RP}^3$ where x (resp. y) is nonzero.
   (a) Prove that Z is a 2-manifold.
   (b) Now we start to cover Z with charts. Let $\varphi_1\colon Z \cap U_x \to \mathbb{R}^2$ be the map sending $[1 : y : z : w] \mapsto (z, w)$. Show that φ_1 is a diffeomorphism.
   (c) The same exact argument should work for the map $\varphi_2\colon Z \cap U_y \to \mathbb{R}^2$ which sends $[x : 1 : z : w] \mapsto (z, w)$. Do these two charts cover all of Z?
   (d) What is the transition function between these two charts?
   (e) (Extra credit) Prove that Z is diffeomorphic to a familiar 2-manifold.

3. Let $M_n \cong \mathbb{R}^{n^2}$ denote the space of all $n \times n$ real matrices, and let
$$X = \{(A, B) \in M_n \times M_n \mid \operatorname{tr}(AB) = 1\}.$$
   Recall that the trace of a square matrix is the sum of the diagonal entries.
   (a) Prove that X is a manifold of dimension $2n^2 - 1$. Prove that
$$T_{(A,B)}X = \{(P, Q) \in M_n \times M_n \mid \operatorname{tr}(AQ) + \operatorname{tr}(BP) = 0\}.$$
   (b) (*) Prove that, for any nonzero matrix A, the set $X_A = \{B \in M_n \mid \operatorname{tr}(AB) = 1\}$ is a manifold (of what dimension). (Hint: find a map for which X_A is the fiber, and prove the map is a submersion.)

4. Let $X = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$.
   (a) Prove that $X \cong \mathbb{R}$. Is $I_2(X, X)$ well-defined? Why or why not?

(b) Let $\overline{X} = \{[x : y : z] \in \mathbb{RP}^2 \mid zy = x^2\}$ be the projective closure of X. Is $I_2(\overline{X}, \overline{X})$ well-defined? Why or why not?

(c) Let $\overline{Y} = \{[x : y : z] \in \mathbb{RP}^2 \mid xy = z^2\}$. Prove that $\overline{X}$ is transverse to $\overline{Y}$ in $\mathbb{RP}^2$. Compute the size of $\overline{X} \cap \overline{Y}$.

(d) Use Ehresmann's theorem to prove that $\overline{X} \cong \overline{Y}$, and that they are homotopic in $\mathbb{RP}^2$. (Hint: it is possible to choose the homotopy to be a straight line in the vector space of degree 2 homogeneous functions.)

5. For $z \in \mathbb{C}$, let $f(z) = \log(3 + \sin(e^{|z|})) + |z|i$. Prove that there is some complex number z for which $f(z) = z^3$.

6. Let f and g be two smooth maps $M \to M$, for an n-manifold M. Prove that $\deg_2(f) \cdot \deg_2(g) = \deg_2(f \circ g)$.

7. (*) A *Lie group* is a group in the category of smooth manifolds. Said more simply, it is a manifold G which is also a group, such that the multiplication map $m\colon G \times G \to G$, $m(g, h) = gh$, and the inverse map $\iota\colon G \to G$, $\iota(g) = g^{-1}$, are both smooth maps.

   (a) Prove that $SL_n(\mathbb{R})$, the set of all $n \times n$ matrices with determinant 1, is a Lie group.

   (b) Prove that left multiplication by $g \in G$ is a diffeomorphism of G. (In particular, you must show it is smooth.)

   (c) Prove that left multiplication by $g \in G$ is homotopic to the identity map, when G is connected.

   (d) Let G be connected, and let $H \subset G$ be a normal subgroup which, in the subspace topology, is discrete. Prove that H is actually in the center $Z(G)$ of G. (Hint: Consider the diffeomorphism of H given by conjugation by a fixed $g \in G$. What is it homotopic to? What can you say about a pair of continuous maps from a discrete space to itself that are homotopic?)

   (e) (Extra credit) Prove that the quotient of a Lie group by a normal discrete subgroup is also a Lie group.