Math 432/532 (Topology), Winter 2016
Midterm Solutions

1. Let $X = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x(x-2)(x+2)\}$.

   (a) Prove that X is a 1-manifold.

   Let $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = y^2 - x(x-2)(x+2)$. Then $df = [-3x^2 + 4, \;\; 2y]$. In order for df to be non-surjective (or equivalently here, nonzero), one must have $y = 0$ and $x = \pm\sqrt{4/3}$. This can't happen anywhere that $f(x, y) = 0$. Thus 0 is a regular value of f, and $f^{-1}(0) = X$ is a 1-manifold.

   (b) For each $(x, y) \in X$, find a basis for $T_{(x,y)}X \subset \mathbb{R}^2$.

   $T_{(x,y)}X = \operatorname{Ker} df$, so it is spanned by the vector $\begin{pmatrix} -2y \\ -3x^2 + 4 \end{pmatrix}$.

   (c) Let $\pi\colon X \to \mathbb{R}$ satisfy $\pi(x, y) = x$. At what points is π a local diffeomorphism?

   By the implicit function theorem, π is a local diffeomorphism where df/dy is nonzero, which is where $y \neq 0$.

   (d) (*) What is the image of π?

   If $x(x+2)(x-2) < 0$ then there is no possible y with $y^2 = x(x+2)(x-2)$. If $x(x+2)(x-2) \geq 0$ then there is such a y. Thus the image is the set of x such that $x(x-2)(x+2) \geq 0$. This happens when $x \geq 2$ or when $-2 \leq x \leq 0$.

   (e) (*) Is X connected? Why or why not?

   The image of a connected set is connected, but $[-2, 0] \cup [2, \infty)$ is not connected, so X is not connected.

2. Let X and Y be manifolds, and $\pi\colon Y \to X$ be a surjective map. We say that $f\colon X \to Y$ is a *section* of π if $\pi \circ f = \mathrm{id}_X$.

   (a) Prove that any section is an immersion.

   Since $\pi \circ f = \mathrm{id}_X$, by the chain rule one has $d\pi \circ df = \mathrm{id}_{T_xX}$ for any point $x \in X$. In particular, df must be injective, or $d\pi \circ df$ could not be injective.

   (b) (*) Prove that any section is an embedding.

   Since $\pi \circ f = \mathrm{id}_X$, f must be injective. For any subset $K \subset Y$, one has $f^{-1}(K) = \pi(K \cap \mathrm{Im\ f})$. (It is not just $\pi(K)$... if K doesn't meet the image of f then $f^{-1}(K)$ is empty, while $\pi(K)$ is certainly not.) We claim that $\mathrm{Im\ f}$ is closed. Then if K is compact, so is $K \cap \mathrm{Im}$, and so is its image under π. Thus f is proper.

   We prove that $\mathrm{Im\ f}$ is closed. Suppose that $x_i \in X$ is a sequence where $f(x_i)$ converges to $y \in Y$. Because π is continuous, $\pi(f(x_i)) = x_i$ converges to $\pi(y)$. But because f is continuous, $f(x_i)$ converges to $f(\pi(y))$. Since a sequence converges to at most one point, $y = f(\pi(y))$ and is in the image of f. Thus any convergent sequence in the image of f converges to something in the image of f, and this image is closed.

   (c) We say that two sections are *transverse* if their images are transverse in Y. If f_1 and f_2 are transverse sections, what can you say about the subspace

   $$\{x \in X \mid f_1(x) = f_2(x)\}?$$

(Is it a manifold? Why or why not? If so, of what dimension?)

Let Z be the image of f_2, a submanifold because any section is an embedding. The subspace above is $f_1^{-1}(Z)$ so by the preimage theorem for transverse maps, it is a manifold of dimension $\dim X + \dim X - \dim Y = 2\dim X - \dim Y$.

(d) Let Y be the Möbius strip and $X = S^1$, and $\pi\colon Y \to X$ be the usual fiber bundle with fiber $(-1, 1)$. I have drawn this on the board. On your paper, draw three sections for which any two are transverse. Make sure I can read your drawing!

Ask me for a drawing.

3. True or false? If true, give a brief sketch why. If false, find a counterexample.

Grads choose 3. Undergrads choose 2. For extra credit, do more and *circle* them (otherwise I grade the first ones I see).

(a) For any smooth map of manifolds $f\colon X \to Y$, its graph $\Gamma_f \subset X \times Y$ is diffeomorphic to X.

True. Let $\pi\colon X \times Y \to X$ be projection. Then f is a section of π.

(b) If f_t, $t \in [0, 1]$, is a smooth family of maps $S^1 \to \mathbb{R}^3$, and f_0 is not an embedding, then there is some $\epsilon > 0$ such that f_t is not an embedding for $t < \epsilon$.

False. Let $g\colon S^1 \to \mathbb{R}^3$ be the usual embedding of S^1 in $\mathbb{R}^2$, followed by the inclusion of $\mathbb{R}^2$ into $\mathbb{R}^3$ as the (xy)-plane. For $t \in \mathbb{R}$, let $f_t(\theta) = tg(\theta)$ be a rescaling of this map. Then f_0 is the constant map at the origin, and is not an embedding, but f_t is an embedding for all $t \neq 0$.

(c) Let X be an n-manifold and Y a m-manifold. If $y \in Y$ is a critical value of a smooth map $f\colon X \to Y$, then $f^{-1}(y)$ is not an $(n - m)$-manifold.

False. Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Then $f^{-1}(0) = \{0\}$ is a 0-manifold.

(d) Let $X \subset \mathbb{R}^N$ be an n-manifold with $n \geq 2$, and $x \in X$. Let v and w be two vectors in T_xX which span a plane. Then there is an embedding of D^2 (the open disk in $\mathbb{R}^2$) into X, which sends the origin to x, and sends the standard basis of T_0D^2 to the vectors v and w.

True. This problem can be attacked locally, so wlog $X = \mathbb{R}^n \subset \mathbb{R}^N$ is a coordinate hyperplane (by the local immersion theorem), and its tangent space can also be identified with $\mathbb{R}^n \subset \mathbb{R}^N$. If A is a linear transformation $\mathbb{R}^2 \to \mathbb{R}^n$ sending the standard basis to v and w, then composing with the inclusion $\mathbb{R}^n \to \mathbb{R}^N$ gives a map $\mathbb{R}^2 \to \mathbb{R}^N$ with the desired property (since $dA = A$ as matrices). Restricting to $D^2 \subset \mathbb{R}^2$ does not change the tangent space at the origin.