Math 316 (Fund. of analysis), Winter 2018
Quiz 2 solutions
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1. **(5 pts)** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and $t \in \mathbb{R}$. State the *sequential* definition for continuity of f at t.

   f is continuous at t if, for every sequence $(x_n)_{n\in\mathbb{N}}$ converging to t, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(t)$.

   It was very common for people to write the ϵ-δ definition of continuity instead. Those who did often had quantifier mistakes - the quantifiers, and their order, are extremely important!!!

Theorem 0.1. Let (a_n) be a sequence with $a_n \geq 0$ for all $n \in \mathbb{N}$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the sequence of partial sums is bounded.

Theorem 0.2. Let (a_n) and (b_n) be sequences satisfying $a_n \geq b_n \geq 0$ for all $n \in \mathbb{N}$. If the series $\sum_{n=1}^{\infty} a_n$ is convergent then the series $\sum_{n=1}^{\infty} b_n$ is convergent.

2. **(7 pts)** *DO ONE:* Prove Theorem 0.1, or prove Theorem 0.2 assuming Theorem 0.1.

   Note: a common and very bad mistake was to mix up the sequence (a_n) and the sequence of its partial sums. If this is your mistake, you should really focus on understanding the difference between sequences and series. I always recommend you set up notation for the sequence of partial sums to avoid confusion!!

   Proof of Theorem 0.1: Let $s_n = \sum_{i=1}^{n} a_i$ be the sequence of partial sums of (a_n). Any convergent sequence is bounded, so if (s_n) converges, then it is bounded. Conversely, suppose (s_n) is bounded. Since $a_n \geq 0$ for all $n \in \mathbb{N}$, (s_n) is increasing. Thus, by the MCT, (s_n) converges.

   Proof of Theorem 0.2: Let $s_n = \sum_{i=1}^{n} a_i$ and $t_n = \sum_{i=1}^{n} b_i$ be the sequences of partial sums. If (s_n) converges, then by Theorem 0.1, (s_n) is bounded. But $a_n \geq b_n \geq 0$ implies that $s_n \geq t_n \geq 0$ for all $n \in \mathbb{N}$, so that the bound for (s_n) is also a bound for (t_n). Hence (t_n) converges, by Theorem 0.1.

   Note: If you didn't use Theorem 0.1 in your proof of Theorem 0.2, you know you're probably doing it wrong, given the way the question was asked.

3. **(6 pts)** For each of the functions defined below: is it continuous at 0? What is the functional limit at 0, or is it undefined? No justification required.

   (a) $f(x) = x$ if $x \in \mathbb{Q}$, and $f(x) = -x$ if $x \notin \mathbb{Q}$.

   (b) $g(x) = x + 1$ if $x \geq 0$, and $g(x) = -x$ if $x < 0$.

   (c) $h(x) = x + 1$ if $x \neq 0$, and $h(0) = -3$.

   f is continuous at 0, and the functional limit is 0.

   g is not continuous at 0, and the functional limit is not defined.

   h is not continuous at 0, but the functional limit is equal to 1.

4. **(12 pts)** Let $f \colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 1$. Using the ϵ-δ definition of continuity, prove that f is continuous everywhere.

   Let us choose an arbitrary $c \in \mathbb{R}$, and prove continuity at c . For any $x \in \mathbb{R}$ we have

   $$|f(x) - f(c)| = |x^2 - 1 - (c^2 - 1)| = |x^2 - c^2| = |x - c| \cdot |x + c|.$$

   We should bound $|x + c|$ above (using a bound which is independent of x). Whenever $|x - c| < 12$, we have $|x + c| \leq |x - c| + |2c| < 2|c| + 12$.

   So let $\epsilon > 0$ be arbitrary, and let $\delta = \min\{12, \frac{\epsilon}{2|c|+12}\}$. For any $x \in \mathbb{R}$, if $|x - c| < \delta$ then

   $$|f(x) - f(c)| = |x - c| \cdot |x + c| < \delta(2|c| + 12) < \epsilon.$$

   Thus f is continuous at c. Since c was arbitrary, f is continuous everywhere.

   Common mistakes:

   - Your chosen δ can NOT depend on x! This destroys the order of the quantifiers: f is continuous at c if, for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $x \in \mathbb{R}$ with $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$. Note that the $\forall x$ occurs after the $\exists \delta$. That means x is not yet defined when δ is being chosen. Said another way, a single choice of δ must work for all x in its neighborhood. So if you tried $\delta = \frac{\epsilon}{|x+c|}$, this is your mistake.

   - When trying to deal with $|x + c|$, one can think that x is getting close to c, so that $|x + c|$ is getting close to $2|c|$. But getting close to is different from being equal to, or being less than. You can NOT replace $|x + c|$ with $2|c|$. Instead, the argument above says that, when x gets within distance 12 of c, $|x + c|$ gets within distance 12 of $2|c|$. This you can accomplish, by asserting later that $\delta < 12$. (Of course, the number 12 is arbitrary.)

- Rule of thumb: if you compute $|f(x) - f(c)|$ and you don't get something which evaluates to zero when $x = c$, you did something wrong! So catch yourself quickly with this sanity check. If you got some signs wrong and wrote $|f(x) - f(c)| = |x^2 - c^2 - 2|$, any algebra you did after this was probably incorrect and an act of desperation. When $x = c$, $|f(x) - f(c)| = 0$, but $|x^2 - c^2 - 2| = 2$ is not zero. A related observation is that when $|x - c|$ is small, one expects $|f(x) - f(c)|$ to be small, but $|x^2 - c^2 - 2|$ to be close to 2. There is no way for this to be less than ϵ, when ϵ is fairly small.

- Some people computed $|f(x) - f(c)|$ for two arbitrary points x and c, but treated the two points in the same way. This is NOT a good approach! You should be checking continuity at a certain point (say, c), so that c is fixed and x varies. Ultimately, what people who made this mistake were trying to prove is:

  for all $\epsilon > 0$ there exists $\delta > 0$ such that, for all $x, c \in \mathbb{R}$, $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$.

  This is an interesting definition - this is called *uniform continuity*, and we'll learn about it in the next quarter. The difference between this definition and continuity itself is this: in continuity, δ can depend upon ϵ and c (but not x), while in uniform continuity δ can only depend upon ϵ. (see the first bullet point)

  Unfortunately, the function $x^2 - 1$ is NOT uniformly continuous. This is related to the fact that its derivative is unbounded, so the ratio between δ (the rate of change of the input) and ϵ (the rate of change of the output) can not be bounded independent of y.

- If you did something like $|f(x) - L| < \epsilon$, without specifying L, you're certainly doing something weird, and I don't know where this came from.

5. **(20 pts)** *DO FOUR OUT OF FIVE.* For each of the following statements, is it true or false? Justification is required.

(a) If $\sum_{n\in\mathbb{N}} x_n$ converges absolutely, and (y_n) is a bounded sequence, then $\sum_{n\in\mathbb{N}} x_n y_n$ converges.

True. If $|y_n| < M$ for all n, then $0 \leq \sum |x_n y_n| \leq M \sum |x_n|$. By Theorem 0.2, $\sum |x_n y_n|$ converges, so $\sum x_n y_n$ converges absolutely.

(b) The series $1 - \frac{1}{1^2} + \frac{1}{2} - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{3^2} + \ldots$ converges.

False. The sequence is not absolutely convergent, because $\sum \frac{1}{n}$ diverges. For a sequence to be conditionally convergent, both the "positive sum" and the "negative sum" have to diverge. But $\sum \frac{1}{n^2}$ converges.
Note: Many people said this series is "THE SAME" as $\sum \frac{1}{n} - \sum \frac{1}{n^2}$. It is NOT the same. See the note to part d.

(c) If $\sum_{n\in\mathbb{N}} x_n$ converges, then $\sum_{n\in\mathbb{N}} (-1)^n x_n$ converges.

False. If $x_n = (-1)^n \frac{1}{n}$, then $\sum x_n$ converges by the alternating series test, but $\sum (-1)^n x_n = \sum \frac{1}{n}$ is divergent.

(d) The following two series have the same limit.

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots, \qquad 1 + \frac{1}{3^2} - \frac{1}{2^2} + \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{4^2} + \ldots$$

True. Any rearrangement of an absolutely convergent series has the same limit. The series $\sum \frac{1}{n^2}$ converges, so these series absolutely converge.

Note: Many people said these series were "THE SAME." They are NOT the same. They are rearrangements. Rearrangements need not have the same limit, unless the sequence is absolutely convergent!!

(e) For $f, g \colon \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$, if f is continuous at x and g is not continuous at x, then $f \cdot g$ is not continuous at x.

Note: Many people thought that $f \cdot g$ meant the composition, but it is the product. $f \circ g$ is the composition. I didn't take off any points for this. Regardless, the answer is the same for both, and I'll give an answer that works for both.

False. Let $f(x) = 0$ for all x, and g be any non-continuous function. Then $f \cdot g(x) = 0$ for all x.

Extra credit problems:

1) The series $\sum_{n\in\mathbb{N}} \frac{1}{n\log(n)}$ diverges.

Proof: We use the Cauchy condensation test. Letting $b_n = \frac{1}{n\log(n)}$, we have

$$2^n b_{2^n} = \frac{2^n}{2^n \log(2^n)} = \frac{1}{n\log(2)}.$$

Thus

$$\sum 2^n b_{2^n} = \frac{1}{\log(2)} \sum \frac{1}{n},$$

which diverges. Hence $\sum b_n$ diverges.

2) Use the ϵ-δ definition of continuity to prove that $f(x) = \frac{x-1}{x}$ is continuous at x, for any $x > 0$.

Proof: Fix $x > 0$. For any $y \in \mathbb{R}$ we have

$$|f(x) - f(y)| = |\frac{x-1}{x} - \frac{y-1}{y}| = |\frac{x-y}{xy}|.$$

We need to bound $\frac{1}{|xy|}$ above. When $|x-y| < x/2$, we have that $y > x/2$, so that $\frac{1}{xy} < \frac{2}{x^2}$.

Fix $\epsilon > 0$, and let $\delta = \min\{x/2, x^2\epsilon/2\}$. Then for any $y \in \mathbb{R}$ with $|x-y| < \delta$, we have

$$|f(x) - f(y)| = |\frac{x-y}{xy}| < \delta\frac{2}{x^2} < \epsilon.$$

Thus f is continuous at x.